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بسم الله الرحمن الرحيم

Estimation of the Parameters of Downton's Bivariate Exponential Distribution Using Moving Extreme Ranked Set Sampling

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بسم الله الرحمن الرحيم

قال الله تعالى :

قُلْ إِنَّ صَلَاتِي وَنُسُكِي وَمَحْيَايَ وَمَمَاتِي لِلَّهِ رَبِّ الْعَالَمِينَ ﴿١٦٢﴾

سورة الانعام

عن ابي هريرة ، رضي الله عنه ، ان رسول الله صلى الله عليه وسلم ، قال :

(ومن سلك طريقاً يلتمس فيه علماً ، سهل الله له به طريقاً إلى الجنة)

(رواه البخاري)

الإهداء :-

إلى نبع الحنان على هذه الأرض ... إلى من الجنة تحت أقدامها ...

إلى أمي ..

إلى من كان لي السند والمعين في كل خطوة أخطوها ...

إلى والدي ..

إلى رمز الصدق والطيبة ...

إلى إخواني وأخواتي ..

إلى كل من ساهم في إتمامه وإنجازه ..

إلى كل مسلم ومسلمة ..

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Hanandeh, Ahmad Ali. Estimation of the Parameters of Downton's Bivariate Exponential Distribution Using Moving Extreme Ranked Set Sampling. Master of Science Thesis, Department of Statistics, Yarmouk University, 2011 (Supervisor: Prof. Mohammad Fraiwan Al-Saleh)

The purpose of this thesis is to estimate the parameters of Downton's bivariate exponential distribution (DBED), using moving extreme ranked set sampling technique (MERSS). The estimators obtained using MERSS are compared via their biases and mean square errors (MSE's) to their counterparts using simple random sampling (SRS). Simulation is used whenever analytical comparison is not possible. It is shown that these estimators when using MERSS with concomitant variable are more efficient than the corresponding ones using SRS. Also, MERSS with concomitant variable is easier to use in practice than RSS with concomitant variable. In addition, we derive the best linear unbiased estimators (BLUE) of some parameters. It is shown that these estimators are very close to the corresponding naive estimators. Moreover, Fisher information matrix of Downton's bivariate exponential distribution is derived and used to find the asymptotic efficiency of the maximum likelihood estimator (MLE) of each of the parameters using MERSS with respect to those based on SRS. It is shown that some of the estimators obtained using MLE based on MERSS are asymptotically more efficient than the corresponding ones based on SRS.

Key Words: Downton's Bivariate Exponential Distribution; Simple Random Sampling; Moving Extreme Ranked Set Sampling; Concomitant Variable; Best Linear Unbiased Estimator; Maximum Likelihood Estimation; Fisher Information Matrix.

الملخص

هناندة، أحـــمد علـــي. تقدير معلمات داونتون الثنائي الأسي بطريقة المعاينة المرتبة المتحركة. رسالة ماجستير في العلوم، قســـــم الاحصـــــــاء، جامعـــــة اليرمـــــوك، 2011.

(المشرف: الأستاذ الدكتور محمد فريوان الصالح).

الهدف من هذه الرسالة هو تقدير معلمات داونتون الثنائي الأسي باستخدام طريقة المعاينة المرتبة المتحركة بأخذ القيم المتطرفة (واحدة من مشتقات المعاينة المرتبة التي تقدم بها العودات والصالح (2001))، بخلاف المعاينة المرتبة، تسمح لنا هذه طريقة بزيادة حجم العينة دون إحداث المزيد من الأخطاء في الترتيب، وقد تمت مقارنة التقديرات التي تم الحصول عليها باستخدام هذه الطريقة مع نظيراتها من التقديرات باستخدام العينة العشوائية البسيطة من خلال التحيز ومتوسط مربعات الأخطاء. وقد استخدمنا المحاكاة كلما كانت المقارنة التحليلية غير ممكنة. وقد ظهر أن التقديرات الناتجة باستخدام هذه الطريقة أكثر كفاءة من تلك الناتجة باستخدام العينة العشوائية البسيطة. كذلك فإن المعاينة المرتبة المتحركة بأخذ القيم المتطرفة هي أسهل عملياً في الإستخدام من العينة المرتبة. وبالإضافة إلى ذلك، قمنا باشتقاق أفضل التقديرات الخطية غير المتحيزة لبعض المعلمات، وقد ظهر أن هذه التقديرات قريبة جداً من التقديرات الأصلية. وعلاوة على ذلك، مصفوفة فيشر لهذا التوزيع حُسبت، واستخدمناها للعثور على مستوى الكفاءة للتقديرات الإحتمالية الأقصى عندما يقترب حجم العينة من المالا نهاية، وقد ظهر أن بعض التقديرات الناتجة باستخدام هذه الطريقة أكثر كفاءة من تلك الناتجة باستخدام العينة العشوائية البسيطة.

CHAPTER ONE

INTRODUCTION AND LITERATURE REVIEW

1. INTRODUCTION

Statistics is the science which is concerned with the collection of data from a population, summarizing and describing data, and ultimately analyzing them to draw inferences about the population characteristics. There are two ways to study the characteristics of any population, if the entire population is sufficiently small, then the researcher can include the entire population in the study. This type of research is called "census". Usually, the population is too large for the researcher to attempt to survey all of its units. A small, but carefully chosen sample can be used. The sample should reflect the characteristics of the population from which it is drawn. In order to make any generalizations about a population, a sample must be representative of the population *i.e.*, a sample resulting from a sampling method that can be expected to adequately reflects the properties of interest of the parent population. A representative sample may be obtained using one or more of several sampling methods. The choice among them depends upon the objectives of the survey and the characteristics of the population.

A short description of some well known sampling techniques is given in Section 2. Our purpose in this thesis is to use moving extreme ranked set sampling (MERSS) to estimate the parameters of Downton's bivariate exponential distribution (DBED). The related literature is reviewed in Section 3. Thesis organization is given in Section 4.

2. SAMPLING TECHNIQUES

In this section we will explain some sampling techniques that are related to our work.

2.1 Some Sampling Techniques

The most popular sampling method that is usually used in statistical studies is **simple random sampling (SRS).** In SRS, the units of the sample are selected randomly from the population. A SRS of size n from a population is a subset of the population consisting of n units selected in such a way that all subsets of size n are equally likely to be selected; but for an infinite population (which we are interested in) each unit of the sample are selected independently and comes from the same population (*iid*). Simple random sampling is the basic building block and point of reference for all other sampling methods. **Stratified random sampling** is another sampling technique. Here, the population is first divided into non-overlapping groups of elements called strata according to some characteristic. Then, a simple random sample is taken from each stratum; within a stratum, all units have equal chances of selection. The chance of selection may vary among strata. **Systematic random sampling** is a third method which can be used in the case of moving population, a random starting element is chosen from the first k elements in the frame using a random number generator. The sample is chosen by going through the population sequentially; *i.e.*, every k^{th} element thereafter. This is known as 1-in-k systematic sample. Another popular random technique is **cluster random sampling.** Here, the population consists of clusters (groups). Then, a simple random sample of clusters is chosen and all units in the chosen clusters are measured.

Recently, attention is being paid to another technique called the ranked set sampling. This technique and some of its variations are the content of the next section. We propose here to apply one variation of the technique namely, moving extreme RSS to the Downton's bivariate exponential distribution.

2.2 Ranked Set Sampling

Seeking to improve the accuracy of crop yield estimates without increasing the number of observations that need to be quantified, **McIntyre (1952)** suggested a sampling technique which is later called RSS. This technique of data collection was introduced for situations where taking the actual measurements on sample observations is difficult (*i.e.*, costly, time-consuming) as compared to the judgment ranking of them.

The ranked set sampling technique can be executed as follows:

Step 1: Randomly draw m simple random samples each of size m from the population of interest.

Step 2: Within each of the m sets, the sampled items are ranked from lowest to largest according to the variable of interest based on the researcher's judgment or by any negligible cost method that does not require actual quantifications.

Step 3: From the first set of m units, the unit ranked lowest is measured. From the second set of m units, the unit ranked second lowest is measured. The process is continued until the m^{th} ranked unit is measured from the m^{th} set. Note that although m^2 units are sampled initially, only m of them are measured with respect to the variable of interest.

The above procedure describes one cycle of the RSS technique.

Step 4: Repeat teps (1-3), if necessary, k independent times (cycles) to obtain a total sample of size $n= mk$ units.

In McIntyre's RSS procedure, it is assumed that the researcher could order a set of size m units with respect to the characteristic of interest. Many authors recommended that m should be 2, 3 or 4 to minimize the ranking error [see **Takahasi and Wakimoto (1968)**].

2.3 Moving Extreme Ranked Set Sampling

There are several variations of RSS; one of them is moving extreme ranked set sampling. In this procedure only the maximum (or minimum) of sets of varied size is identified (by judgment) for quantification. The MERSS, as described by **Al-Odat and Al-Saleh (2001)** and **Al-Saleh and Al-Hadrami (2003 a, b)**, can be executed as follows:-

Step 1: Select m simple random samples of size 1, 2, 3, ..., m, respectively.

Step 2: For each of these samples, measure accurately the maximum ordered observation from the first set identified by judgment, the maximum ordered observation from the second set, *etc.* The process continues in this way until the maximum ordered observation from the last m^{th} sample is measured.

Step 3: Repeated steps 1 and 2, if necessary, r times to obtain a sample of size $n = rm$.

3. Literature Review

This review of literature is based on **Kaur et al. (1995).** Recent articles that come after that date are also reviewed. Only research directly related to our proposed work will be given in some details.

Ranked set sampling was introduced by **McIntyre (1952)**, in the context of estimating pasture yields. He claimed, without providing a mathematical proof, the following:-

1. The mean of the RSS, $\hat{\mu}_{RSS} = \frac{1}{mk}\sum_{r=1}^{k}\sum_{i=1}^{m} X_{[i]r}$, regardless of any error in judgment ranking, is an unbiased estimator of the population mean (μ), where $X_{[i]r}$ is the i^{th} judgment order statistic in the r^{th} iteration.
2. With perfect ranking, the efficiency of RSS *w.r.t.* SRS in estimation the population mean is nearly $\frac{m+1}{2}$ for typical unimodal distributions.

3. The efficiency of the estimators of higher population moments based on RSS are only slightly better than those based on SRS.

Takahasi and Wakimoto (1968) obtained the following main theoretical results:-

Under perfect ranking, the mean of a ranked set sample is an unbiased estimator of the population mean, and its variance is always smaller than the variance of the mean of a simple random sample of equal size. Also, they showed that:

$$f(x)=\frac{1}{m}\sum_{i=1}^{m} f_i(x),\quad \mu=\frac{1}{m}\sum_{i=1}^{m}\mu_i \text{, and } \sigma^2=\frac{1}{m}\sum_{i=1}^{m}\sigma_i^2+\frac{1}{m}\sum_{i=1}^{m}(\mu_i-\mu)^2,$$

where, $f(x)$ is the *pdf* of a *r.v.* X, $\mu = E(X)$,

$\sigma^2 = Var(X)$,

$f_i(x)$ is the *pdf* of the i^{th} order statistic, $\mu_i = E(X_{(i)})$, and

$\sigma_i^2 = Var(X_{(i)})$.

Comparative performance of the estimators is assessed using either the relative precision (*RP*) (efficiency), or relative saving (*RS*), which are defined as follows:

$$eff(\hat{\mu}_{RSS},\hat{\mu}_{SRS}) = RP = \frac{\operatorname{var}(\hat{\mu}_{SRS})}{\operatorname{var}(\hat{\mu}_{RSS})},$$

$$\text{and } RS = \frac{\operatorname{var}(\hat{\mu}_{SRS})-\operatorname{var}(\hat{\mu}_{RSS})}{\operatorname{var}(\hat{\mu}_{SRS})} = 1-\frac{1}{RP}.$$

They also showed that:

$$\operatorname{var}(\hat{\mu}_{RSS}) = \frac{1}{m}\left(\sigma^2-\frac{1}{m}\sum_{i=1}^{m}(\mu_i-\mu)^2\right),$$

$$RS=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\mu_i-\mu}{\sigma}\right)^2,\quad 0\le RS\le\frac{m-1}{m+1} \quad\text{and}\quad 1\le eff(\hat{\mu}_{RSS},\hat{\mu}_{SRS})\le\frac{m+1}{2}.$$

"The lower bound is attained if and only if the parent distribution is degenerate. The upper bound is attained if and only if the parent distribution is rectangular". [Kaur et al. (1995)]

Stokes (1977) studied RSS with concomitant variables. She assumed that each sampling unit has a bivariate response (X,Y), where X is the variable of interest and Y is the concomitant variable that is not of direct interest but is relatively easy to measure.

Samawi et al. (1996) introduced extreme ranked set sampling (ERSS) procedure. It was shown that the ERSS estimator of the mean is more efficient than the usual SRS mean and unbiased if the underlying distribution is symmetric. In this ERSS, only the two extremes (Min, Max) are identified by judgment for different sets.

Al-Saleh and Al-Kadiri (2000) considered double RSS (DRSS) as a procedure that increase the efficiency of RSS estimator without increasing the set size m. It was shown that the DRSS estimator of the mean is more efficient than that using RSS. Furthermore, ranking in the second stage is in some sense, easier than ranking in the first stage.

Al-Saleh and Al-Omari (2002) generalized DRSS to multistage ranked set sampling. They showed that the efficiency is always between 1 and m^2 for all distributions and equal to m^2 for the uniform distributions, when the number of stages goes to infinity. See also **Al-Saleh and Samuh (2008) and Samuh and Al-Saleh (2011).**

Al-Saleh and Zheng (2002) proposed a new RSS for two characteristics and called it a bivariate ranked set sampling.

Bayesian estimation with RSS was considered by **Al-Saleh and Muttlak (2000)** and **Al-Saleh and Abu-Hawwas (2002).**

Moving extremes ranked set sampling is a useful modification of ranked set sampling. Unlike RSS, MERSS allows for an increase of set size without introducing too much ranking error. MERSS was introduced by **Al-Odat and Al-Saleh (2001)**, they introduced the concept of varied set size RSS.

Al-Saleh and Al-Hadrami (2003 a, b) used varied set size of RSS (coined by them MERSS) for estimating the mean of the normal and exponential distributions, and they showed that this procedure could be more useful than SRS for estimating the mean of symmetric distributions.

Ananbeh (2004) (see also Al-Saleh and Ananbeh (2005, 2006)) estimated the means and the correlation of the bivariate normal distribution using MERSS with concomitant variable.

Al-Saleh and Samawi (2010) estimated the odds based on MERSS. The suggested estimator based on MERSS is motivated by some of the theoretical properties of the sum of geometric series.

Ranked set sampling and some of its variation were used by many authors in parametric estimation: bivariate normal, exponential, Downton's bivariate exponential,..., *etc*.

Moran (1967) introduced a bivariate exponential distribution. Many authors have considered the reliability of theoretical derivations with bivariate exponential distributions.

One of the most important bivariate distributions in reliability theory is the bivariate exponential; there are various bivariate exponential distributions. In this research, we are interested in Downton's bivariate exponential distribution with probability density function (*pdf*):

$$f(x,y;\lambda_1,\lambda_2,\rho)=\frac{1}{\lambda_1\lambda_2(1-\rho)}\exp\left[-\left(\frac{x}{\lambda_1(1-\rho)}+\frac{y}{\lambda_2(1-\rho)}\right)\right]\times I_0\left[\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right] \quad (1)$$

where $x,y,\lambda_1,\lambda_2>0$, $0\le\rho<1$ and $I_0(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!^2}$ is the modified Bessel function of the first kind of order zero.

Downton (1970) noted that "for (1) to be a density function the correlation ρ must be positive, a restriction which arises naturally from the model by which the density was obtained In an equipment failure context it is difficult to imagine a situation which could lead to negative correlation"

Let (X,Y) be a random vector from (1.1), then the marginal distributions of X and Y are exponential with parameters λ_1 and λ_2, respectively; so, in particular, $E(X)=\lambda_1$ and $E(Y)=\lambda_2$.

Downton (1970) showed that:

$$E(Y\mid X=x)=(1-\rho)\lambda_2+\rho\frac{\lambda_2}{\lambda_1}x,$$

and

$$Var(Y\mid X=x)=(1-\rho)^2\lambda_2^2+2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1}x.$$

The parameter ρ is the correlation coefficient between X and Y with independence corresponding to $\rho=0$; since $I_0(0)=1$. Also $0\le\rho<1$.

This distribution is a candidate distribution for positively correlated bivariate exponential data, in which the conditional mean and variance of one variable is increasing function of the other variable.

This distribution has received real applications in several fields. The following are some of them:

- ✓ **Lefebvre (2004)** considered forecasting the flow values of the Mistassibi River in Quebec (Canada).
- ✓ Oil pollution of sea water and the tar deposit near sea shore. Here, X represents the tar deposit and Y represents the oil pollution and the two variables are highly positively related. In this application, the oil pollution is hard and expensive to measure while the tar deposit can be ranked visually **(Bain (1978), Chacko and Thomas (2008))**.
- ✓ **Kim and Rao (2000)** considered a two-dimensional warranty that is offered for new automobiles. Here, the warranty is valid until either a pre-specific time limit or pre-specific usage limit (in miles driven) is exceeded.
- ✓ **Choo and Conolly (1979)** used this distribution in queueing systems.
- ✓ **Nagao and Kadoya (1971)** suggested that this distribution can be used for such pairs of hydrological quantities as a streamflow at two points on a river or rainfall at two locations.
- ✓ **Cordova and Rodriguez-Iturbe (1985)** considered it as a model of the intensity and duration of a storm of rainfall.
- ✓ **Reliability.** A model for joint density of failure times when "shocks" are causing different types of failure to components **(Nadarajah and Kotz (2006))**.

For more application see also Balakrishna, N. and Lai, C. D. (2009) pages 401 - 466.

The above density was derived in a different form by **Moran (1967)**. The above form of the density with conditional expectation and variance was derived by **Downton (1970)**. It is a special case of **Kibble's (1941)** bivariate gamma distribution. Note that unlike the bivariate normal distribution, the conditional variance is not fixed in x. Other results about this distribution can be found in **Kotz et al. (2000)**.

Iliopoulos (2003) noticed that the joint sufficient statistic is $(X_1Y_1, \ldots, X_mY_m, \sum_{i=1}^{m} X_i, \sum_{i=1}^{m} Y_i)$

Consider a random vector (X,Y) from DBED. **Nagao and Kadoya (1971)** showed that the maximum likelihood estimators of λ_1 and λ_2 are $\overline{X}$ and $\overline{Y}$, respectively.

Two classes of estimators of ρ based on the complete bivariate samples were derived by **Al-Saadi and Young (1980)**:

(i) Method of moments estimators based on the statistic $\frac{\sum_{i=1}^{m} X_iY_i}{m\overline{X}\overline{Y}}$, where

$$\overline{X} = \frac{\sum_{i=1}^{m} X_i}{m} \text{ and } \overline{Y} = \frac{\sum_{i=1}^{m} Y_i}{m}.$$

They suggested that:

$$\hat{\rho} = \frac{\sum_{i=1}^{m} X_iY_i}{m\overline{X}\overline{Y}} - 1$$

as an estimator of ρ, and using the condition that $0 \le \rho < 1$, a modified estimator for ρ is:

$$\hat{\rho}_1 = \begin{cases} 0 & if \quad \hat{\rho} < 0 \\ \hat{\rho} & if \quad 0 \le \hat{\rho} \le 1 \\ 1 & if \quad \hat{\rho} > 1 \end{cases}$$

(ii) Estimators based on the sample correlation coefficient (r)

$$r = \frac{\sum_{i=1}^{m}(X_i - \overline{X})(Y_i - \overline{Y})}{\sqrt{\sum_{i=1}^{m}(X_i - \overline{X})^2 \sum_{i=1}^{m}(Y_i - \overline{Y})^2}}.$$

Using the condition that $0 \le \rho < 1$, they suggested that:

$$\hat{\rho}_2 = \begin{cases} 0 & if \quad -1 \le r < 0 \\ r & if \quad r \ge 0 \end{cases}$$

Note that the first estimator in (*i*) is a function of the sufficient statistic, while the estimator in (*ii*) is not, so we will use the first estimator.

Al-Saleh and Diab (2009) estimated the parameters of Downton's bivariate exponential distribution based on a ranked set sample. Parametric and nonparametric methods were considered. The suggested estimators were compared to the corresponding ones based on simple random sampling. They noticed that some of the suggested estimators are significantly more efficient than the ones based on simple random sampling.

He and Nagaraja (2011) estimated the correlation of Downton's bivariate exponential distribution when all other parameters are unknown using incomplete samples made from (*i*) all the Y-values and the ranks of associated X-values *i.e.*, $(i, Y_{[i:n]})$, $1 \le i \le n$, (*ii*) a Type II right-censored bivariate sample consisting of $(X_{i:n}, Y_{[i:n]})$, $1 \le i \le r < n$, in both cases using simulation, they found that the preferred estimator under (*i*) is a function of the sample correlation of $(X_{i:n}, Y_{[i:n]})$ values, and under (*ii*), a method of moments estimator involving the regression function is preferred.

For more details about RSS and its variations, see also **Al-Saleh and Endeer (2007), Al-Saleh (2006), Sinha (2005), Sroka et al. (2005), Wolfe (2004), Chen et al. (2004)** and **Zheng and Al-Saleh (2002).**

4. THESIS ORGNIZATION

The coming chapters in this thesis are organized as follows:

Chapter 2 dealt with the estimation of λ_1 and λ_2 using SRS and MERSS for the two cases of known and unknown ρ, also, estimation of the correlation coefficient ρ using SRS and MERSS for the two cases of known and unknown λ_1 and λ_2 is discussed. We also derived the best linear unbiased estimation of λ_1 and λ_2 using SRS and MERSS. The efficiency of these estimators are also obtained.

Chapter 3 dealt with the estimation of λ_1, λ_2 and ρ using the method of maximum likelihood estimation based on SRS and MERSS. The asymptotic efficiency of these estimators are also obtained.

Conclusions and some suggested further work are given in Chapter 4.

CHAPTER TWO

Estimation of the Parameters of Downton's Bivariate Exponential Distribution Using Moving Extreme Ranked Set Sampling

1. Introduction

In this chapter, we consider the estimation of the parameters of DBED $(\lambda_1, \lambda_2 \,\&\, \rho)$, using MERSS. The suggested estimators are compared with the corresponding ones using SRS. Estimation of λ_1 and λ_2 for the two cases of known and unknown correlation coefficient ρ is considered. Furthermore, estimation of the correlation coefficient ρ for the two cases of known and unknown λ_1 and λ_2 is also discussed.

2. Expectations and Variances of $X_{(k:k)}$ and $Y_{[k:k]}$

Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_m,Y_m)$, $m \geq 2$, be a random sample from the Downton's bivariate exponential distribution, DBED$(\lambda_1,\lambda_2,\rho)$, with common *pdf*:

$$f(x,y;\lambda_1,\lambda_2,\rho)=\frac{1}{\lambda_1\lambda_2(1-\rho)}\exp\left[-\left(\frac{x}{\lambda_1(1-\rho)}+\frac{y}{\lambda_2(1-\rho)}\right)\right]\times I_0\left[\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right] \quad (1)$$

where, $x,y,\lambda_1,\lambda_2>0$, $0\leq\rho<1$ and $I_0(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!^2}$ is the modified Bessel function of the first kind of order zero.

Let $f_X(x)$, $f_Y(y)$ be the marginal density of X and Y, respectively, and let $F_X(x)$, $F_Y(y)$ be the corresponding cumulative distributions of X and Y, respectively. It can be verified that the marginal density of X and Y are exponential with parameters λ_1 and λ_2, respectively; so, in particular, $E(X)=\lambda_1$ and $E(Y)=\lambda_2$. (see **Ilipoulos (2003)).**

The marginal probability density functions can be written as:

$$f_X(x)=\frac{1}{\lambda_1}\exp\left(-\frac{x}{\lambda_1}\right);x>0,$$

and

$$f_Y(y)=\frac{1}{\lambda_2}\exp\left(-\frac{y}{\lambda_2}\right);y>0.$$

Also, the marginal distribution functions are:

$$F_X(x)=1-\exp\left(-\frac{x}{\lambda_1}\right)$$

and

$$F_Y(y)=1-\exp\left(-\frac{y}{\lambda_2}\right).$$

Now, suppose that $\{(X_{(1:1)},Y_{[1:1]}),(X_{(2:2)},Y_{[2:2]})...,(X_{(k:k)},Y_{[k:k]})\}$, $k=1,2,\ldots,m$, be a MERSS sample from DBED, $X_{(i:m)}$ is the i^{th} order statistic of $X_1,X_2,...,X_m$ and $Y_{[i:m]}$ is the Y-variates paired with $X_{(i:k)}$; $Y_{[i:k]}$ is called a concomitant order statistic. If the judgment ranking on X-variates is perfect, then for $k=1,2,\ldots,m$, $(X_{(k:k)},Y_{[k:k]})$ has the density $f_{X_{(k:k)},Y_{[k:k]}}(x,y)$ by:

$$f_{X_{(k:k)},Y_{[k:k]}}(x,y)=f_{X_{(k:k)}}(x)f_{Y|X}(y\,|\,x),\text{(see Yang (1977))} \tag{2}$$

where, $f_{X_{(k:k)}}(x)$ is the density of the k^{th} order statistic of a SRS of size k from an exponential distribution. $f_{Y_{[k:k]}}(x)$ is the density of the corresponding induced rank of Y, and $f_{Y|X}(y\,|\,x)$ is the conditional pdf of $(Y\,|\,X=x)$.

But,

$$f_{X_{(k:k)}}(x)=k(F_X(x))^{k-1}f_X(x)$$

$$=k(1-\exp\left(-\frac{x}{\lambda_1}\right))^{k-1}f_X(x). \tag{3}$$

Also,

$$f_{X_{(k:k)},Y_{[k:k]}}(x,y)=f_{Y|X}(y\,|\,X=x)f_{X_{(k:k)}}(x)$$

$$= \frac{f_{X,Y}(x,y)}{f_X(x)} f_{X_{(k:k)}}(x).$$

Applying (3), we get

$$f_{X_{(k:k)},Y_{[k:k]}}(x,y) = \frac{f_{X,Y}(x,y)}{f_X(x)} k(1-\exp\left(-\frac{x}{\lambda_1}\right))^{k-1} f_X(x),$$

$$= k(1-\exp\left(-\frac{x}{\lambda_1}\right))^{k-1} f_{X,Y}(x,y). \qquad (4)$$

It can be verified that

$$X_{(k:k)} \overset{d}{=} \sum_{j=1}^{k} \frac{Z_j}{k-j+1},$$ where $Z_1, Z_2, \ldots, Z_k$ are *iid* $\exp(\lambda_1)$ (see **Yang (1977)**).

It follows that

$$E(X_{(k:k)}) = \sum_{j=1}^{k} \frac{E(Z_j)}{k-j+1} = \lambda_1 \sum_{j=1}^{k} \frac{1}{k-j+1}. \qquad (5)$$

Also,

$$Var(X_{(k:k)}) = \sum_{j=1}^{k} \frac{Var(Z_j)}{(k-j+1)^2} = \lambda_1^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2}. \qquad (6)$$

Now,

$$E(Y_{[k:k]}) = E(E(Y_{[k:k]} \mid X_{(k:k)})).$$

But

$$E(Y \mid X = x) = (1-\rho)\lambda_2 + \rho \frac{\lambda_2}{\lambda_1} x.$$

Also, by **Downton (1970)** and **Yang (1977)**

$$Y_{[k:k]} \mid X_{(k:k)} \overset{d}{=} Y \mid X.$$

It follows that

$$E(Y_{[k:k]}) = E(E(Y_{[k:k]} \mid X_{(k:k)})) = E(E(Y \mid X = X_{(k:k)}))$$

$$= (1-\rho)\lambda_2 + \rho \frac{\lambda_2}{\lambda_1} E(X_{(k:k)}).$$

Applying (5), we get

$$E(Y_{[k:k]}) = (1-\rho)\lambda_2 + \rho\lambda_2 \sum_{j=1}^{k} \frac{1}{k-j+1}. \qquad (7)$$

Also,

$$Var(Y_{[k:k]}) = E(Var(Y_{[k:k]} \mid X_{(k:k)})) + Var(E(Y_{[k:k]} \mid X_{(k:k)})).$$

But,

$$Var(Y_{[k:k]} \mid X_{(k:k)}) = Var(Y \mid X = X_{(k:k)}) = (1-\rho)^2 \lambda_2^2 + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1} X_{(k:k)}.$$

It follows that

$$E(Var(Y_{[k:k]} \mid X_{(k:k)})) = (1-\rho)^2 \lambda_2^2 + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1} E(X_{(k:k)}).$$

Applying (5), we get

$$E(Var(Y_{[k:k]} \mid X_{(k:k)})) = (1-\rho)^2 \lambda_2^2 + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1}\left(\lambda_1 \sum_{j=1}^{k} \frac{1}{k-j+1}\right)$$

$$= \lambda_2^2\left((1-\rho)^2 + 2\rho(1-\rho)\left(\sum_{j=1}^{k} \frac{1}{k-j+1}\right)\right).$$

Also,

$$Var(E(Y_{[k:k]} \mid X_{(k:k)})) = Var\left((1-\rho)\lambda_2 + \rho\frac{\lambda_2}{\lambda_1} X_{(k:k)}\right) = \rho^2 \frac{\lambda_2^2}{\lambda_1^2} Var(X_{(k:k)}).$$

Applying (6), we get

$$Var(E(Y_{[k:k]} \mid X_{(k:k)})) = \rho^2 \lambda_2^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2}.$$

Hence,

$$Var(Y_{[k:k]}) = (1-\rho)\lambda_2^2\left((1-\rho) + 2\rho\sum_{j=1}^{k}\frac{1}{k-j+1}\right) + \rho^2\lambda_2^2\sum_{j=1}^{k}\frac{1}{(k-j+1)^2}$$

$$= \lambda_2^2\left\{(1-\rho)^2 + 2\rho(1-\rho)\sum_{j=1}^{k}\frac{1}{k-j+1} + \rho^2\sum_{j=1}^{k}\frac{1}{(k-j+1)^2}\right\}. \quad (8)$$

The previous theoretical results are summarized in the following theorems.

Theorem (2-1)

Let $\{(X_1,Y_1),(X_2,Y_2)...,(X_m,Y_m)\}$ be a SRS from DBED $(\lambda_1,\lambda_2,\rho)$ and let $\{(X_{(1:1)},Y_{[1:1]}),(X_{(2:2)},Y_{[2:2]})...,(X_{(k:k)},Y_{[k:k]})\}$, $k = 1,2,\ldots,m$, be an MERSS sample taken from the same distribution. Then

a. $E(X_{(k:k)}) = \lambda_1 \sum_{j=1}^{k} \frac{1}{k-j+1}$

b. $Var(X_{(k:k)}) = \lambda_1^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2}$

c. $E(Y_{[k:k]}) = (1-\rho)\lambda_2 + \rho\lambda_2 \sum_{j=1}^{k} \frac{1}{k-j+1}$

d. $Var(Y_{[k:k]}) = \lambda_2^2 \left\{ (1-\rho)^2 + 2\rho(1-\rho) \sum_{j=1}^{k} \frac{1}{k-j+1} + \rho^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2} \right\}$

3. Unbiased Estimation of λ_1 and λ_2 Using MERSS, When ρ is known

In this section we consider the estimation of λ_1 and λ_2 when ρ is known using MERSS and SRS, and find the variances of the suggested estimators. Also, we compare the variances of these estimators.

Let denote the unbiased estimators of λ_1 and λ_2 based on SRS by $\hat{\lambda}_{1\,SRS}$ and $\hat{\lambda}_{2\,SRS}$, respectively, and those based on MERSS by $\hat{\lambda}_{1\,MERSS}$ and $\hat{\lambda}_{2\,MERSS}$, respectively. These estimators can be derived as follows:

$$E(\sum_{k=1}^{m} X_{(k:k)}) = \sum_{k=1}^{m} E(X_{(k:k)}) = \lambda_1 \sum_{k=1}^{m}\sum_{j=1}^{k} \frac{1}{k-j+1}.$$

For simplicity, let

$$S_m = \sum_{k=1}^{m}\sum_{j=1}^{k} \frac{1}{k-j+1}.$$

So,

$$S_1 = 1,\ S_2 = 1 + (1+\frac{1}{2}) = 2.5,\ S_3 = 1 + (1+\frac{1}{2}) + (1+\frac{1}{2}+\frac{1}{3}) = 4.333 \quad ,\ldots,$$

$S_m = 1 + (1+\frac{1}{2}) + (1+\frac{1}{2}+\frac{1}{3}) + \ldots + (1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m})$. It can be easily verified that

$$S_m = m + \frac{m-1}{2} + \frac{m-2}{3} + \ldots + \frac{1}{m};$$

Therefore,

$$S_m = \sum_{k=1}^{m} \frac{m+1-k}{k}$$

So,

$$E(\sum_{k=1}^{m} X_{(k:k)}) = \lambda_1 S_m .$$

Hence,

$$\hat{\lambda}_{1\,MERSS} = \frac{1}{S_m} \sum_{k=1}^{m} X_{(k:k)} \qquad (9)$$

is an unbiased estimator of λ_1.

Now,

$$Var(\hat{\lambda}_{1\,MERSS}) = Var(\frac{1}{S_m} \sum_{k=1}^{m} X_{(k:k)}) = \frac{1}{S_m^2} \sum_{i=1}^{m} Var(X_{(k:k)})$$

$$= \frac{1}{S_m^2} \sum_{i=1}^{m} Var(\sum_{j=1}^{k} \frac{Z_j}{k-j+1}) = \frac{\lambda_1^2}{S_m^2} \sum_{k=1}^{m} \sum_{j=1}^{k} \frac{1}{(k-j+1)^2} .$$

For simplicity, let

$$C_m = \sum_{k=1}^{m} \sum_{j=1}^{k} \frac{1}{(k-j+1)^2} .$$

So,

$$C_1 = 1,\ C_2 = 1 + (1 + \frac{1}{4}) = 2.25,\ C_3 = 1 + (1 + \frac{1}{4}) + (1 + \frac{1}{4} + \frac{1}{9}) = 3.61, \ldots,$$

$$C_m = 1 + (1 + \frac{1}{4}) + (1 + \frac{1}{4} + \frac{1}{9}) + \ldots + (1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{m^2}),$$

$$C_m = m + (\frac{m-1}{4}) + (\frac{m-2}{9}) + \ldots + (\frac{1}{m^2}) = \sum_{k=1}^{m} \frac{(m+1)-k}{k^2}$$

Therefore,

$$C_m = \sum_{k=1}^{m} \frac{(m+1)-k}{k^2}$$

And,

$$Var(\hat{\lambda}_{1\,MERSS}) = \frac{C_m}{S_m^2} \lambda_1^2 . \qquad (10)$$

Let

$$\hat{\lambda}_{1\,SRS} = \frac{1}{m} \sum_{i=1}^{m} X_i .$$

It is well-known that $\hat{\lambda}_{1\,SRS}$ is an unbiased estimator of λ_1 .

Also,

$$Var(\hat{\lambda}_{1SRS}) = Var(\frac{1}{m}\sum_{i=1}^{m} X_i) = \frac{1}{m^2}\sum_{i=1}^{m} Var(X_i) = \frac{m\lambda_1^2}{m^2} = \frac{\lambda_1^2}{m}. \qquad (11)$$

Hence, using (10) and (11) we get

$$Eff(\hat{\lambda}_{1\,MERSS}, \hat{\lambda}_{1\,SRS}) = \frac{Var(\hat{\lambda}_{1\,SRS})}{Var(\hat{\lambda}_{1\,MERSS})} = \frac{\lambda_1^2}{m} \bigg/ \left(\frac{C_m}{S_m^2}\right)\lambda_1^2. = \frac{S_m^2}{mC_m}$$

$$= \frac{\left(\sum_{k=1}^{m}\frac{(m+1)-k}{k}\right)^2}{m\sum_{k=1}^{m}\frac{(m+1)-k}{k^2}} = \frac{\sum_{k=1}^{m}\frac{(m+1)-k}{k}}{m} \times \frac{\sum_{k=1}^{m}\frac{(m+1)-k}{k}}{\sum_{k=1}^{m}\frac{(m+1)-k}{k^2}}$$

$$= \sum_{k=1}^{m}\frac{(m+1)-k}{mk} \times \frac{\sum_{k=1}^{m}\frac{(m+1)-k}{k}}{\sum_{k=1}^{m}\frac{(m+1)-k}{k^2}}, \qquad (12)$$

which is clearly larger than 1 and increasing in m.

The first term $\left\{\sum_{k=1}^{m}\frac{(m+1)-k}{mk}\right\}$ is clearly larger than 1, and the second term $\left\{\frac{\sum_{k=1}^{m}\frac{(m+1)-k}{k}}{\sum_{k=1}^{m}\frac{(m+1)-k}{k^2}}\right\}$ is also larger than 1, since k is positive, and the sum in the numerator is divided by k, but the sum in the denominator is divided by k^2 ,*i.e.*, the sum in numerator is less than the sum in the denominator, so $\frac{S_m^2}{mC_m}$ is lager than 1.

The previous theoretical results are summarized in the following theorems:

Theorem (2-2)

a. $\hat{\lambda}_{1\,MERSS} = \frac{1}{\sum_{k=1}^{m}\frac{(m+1)-k}{k}}\sum_{k=1}^{m} X_{(k:k)}$ is an unbiased estimator of λ_1;

b. $Var(\hat{\lambda}_{1\,MERSS}) = \frac{\sum_{k=1}^{m}\frac{(m+1)-k}{k^2}}{\left[\sum_{k=1}^{m}\frac{(m+1)-k}{k}\right]^2}\lambda_1^2.$

c. $Eff(\hat{\lambda}_{1\ MERSS}, \hat{\lambda}_{1\ SRS}) = \dfrac{\left(\sum_{k=1}^{m} \dfrac{(m+1)-k}{k}\right)^2}{m\sum_{k=1}^{m} \dfrac{(m+1)-k}{k^2}}$ is increasing in the set size m and greater than 1 for $m \geq 2$.

Now,

$$E(\sum_{k=1}^{m} Y_{[k:k]}) = \sum_{k=1}^{m} E(Y_{[k:k]}) = \sum_{k=1}^{m}\left\{(1-\rho)\lambda_2 + \rho\lambda_2 \sum_{j=1}^{k} \frac{1}{k-j+1}\right\}$$

$$= m(1-\rho)\lambda_2 + \rho\lambda_2 \sum_{k=1}^{m}\sum_{j=1}^{k} \frac{1}{k-j+1} = m(1-\rho)\lambda_2 + \rho\lambda_2 S_m$$

$$= \lambda_2\left[m(1-\rho) + \rho S_m\right]$$

Hence, when ρ is known:

$$\hat{\lambda}_{2\ MERSS} = \frac{1}{m(1-\rho)+\rho S_m} \sum_{k=1}^{m} Y_{[k:k]} \text{ is an unbiased estimator of } \lambda_2. \qquad (13)$$

It follows that

$$Var(\hat{\lambda}_{2\ MERSS}) = Var\left\{\frac{1}{m(1-\rho)+\rho S_m} \sum_{k=1}^{m} Y_{[k:k]}\right\} = \frac{1}{(m(1-\rho)+\rho S_m)^2} \sum_{k=1}^{m} Var(Y_{[k:k]})$$

$$= \frac{1}{(m(1-\rho)+\rho S_m)^2} \sum_{k=1}^{m}\left\{(1-\rho)\lambda_2^2\left((1-\rho) + 2\rho \sum_{j=1}^{k} \frac{1}{k-j+1}\right) + \rho^2\lambda_2^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2})\right\}.$$

Then

$$Var(\hat{\lambda}_{2\ MERSS}) = \frac{\lambda_2^2}{(m(1-\rho)+\rho S_m)^2}\left(m(1-\rho)^2 + 2\rho(1-\rho)S_m + \rho^2 C_m\right). \qquad (14)$$

Let

$$\hat{\lambda}_{2\ SRS} = \frac{1}{m}\sum_{i=1}^{m} Y_i.$$

It is well-known that $\hat{\lambda}_{2\ SRS}$ is an unbiased estimator of λ_2.

Also,

$$Var(\hat{\lambda}_{2\ SRS}) = Var(\frac{1}{m}\sum_{i=1}^{m} Y_i) = \frac{1}{m^2}\sum_{i=1}^{m} Var(Y_i) = \frac{m\lambda_2^2}{m^2} = \frac{\lambda_2^2}{m}. \qquad (15)$$

Hence, using (14) and (15), we get

$$Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS}) = \frac{Var(\hat{\lambda}_{2\,SRS})}{Var(\hat{\lambda}_{2\,MERSS})} = \frac{\lambda_2^2}{m} \Bigg/ \frac{\left(m(1-\rho)^2 + 2\rho(1-\rho)S_m + \rho^2 C_m\right)}{\left(m(1-\rho)+\rho S_m\right)^2}\lambda_2^2$$

$$= \frac{\left(m(1-\rho)+\rho S_m\right)^2}{m\left(m(1-\rho)^2 + 2\rho(1-\rho)S_m + \rho^2 C_m\right)}, \qquad (16)$$

which is clearly larger than 1, since

$$Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS}) = \frac{m^2(1-\rho)^2 + 2m\rho(1-\rho)S_m + \rho^2 S_m^2}{m^2(1-\rho)^2 + 2m\rho(1-\rho)S_m + m\rho^2 C_m} \text{ is larger than 1}$$

$\Leftrightarrow \dfrac{S_m^2}{mC_m}$ is larger than 1.

Note that $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS}) \xrightarrow{as\ \ \rho\to 1} Eff(\hat{\lambda}_{1\,MERSS},\hat{\lambda}_{1\,SRS})$, and goes to 1 as $\rho \to 0$ for fixed m.

Table (2.1) contains the efficiency of $\hat{\lambda}_{1\,MERSS}$ and $\hat{\lambda}_{2\,MERSS}$ with respect to the corresponding estimators using SRS.

Table (2.1): The efficiency of $\hat{\lambda}_{1\,MERSS}$ $w.r.t.\ \hat{\lambda}_{1\,SRS}$ ($\hat{\lambda}_{2\,MERSS}$ $w.r.t.\ \hat{\lambda}_{2\,SRS}$)

ρ \ m	2	3	4	5
0.1	1.389(1.004)	1.733(1.008)	2.044(1.012)	2.330(1.015)
0.2	1.389(1.016)	1.733(1.031)	2.044(1.044)	2.330(1.055)
0.3	1.389(1.035)	1.733(1.066)	2.044(1.093)	2.330(1.116)
0.4	1.389(1.061)	1.733(1.113)	2.044(1.158)	2.330(1.197)
0.5	1.389(1.095)	1.733(1.173)	2.044(1.241)	2.330(1.299)
0.6	1.389(1.135)	1.733(1.247)	2.044(1.342)	2.330(1.425)
0.7	1.389(1.184)	1.733(1.336)	2.044(1.467)	2.330(1.581)
0.8	1.389(1.241)	1.733(1.444)	2.044(1.619)	2.330(1.774)
0.9	1.389(1.309)	1.733(1.574)	2.044(1.808)	2.330(2.017)

Based on the previous table we conclude the following:

1. $Eff(\hat{\lambda}_{1\,MERSS},\hat{\lambda}_{1\,SRS})$ and $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS})$ are always greater than 1.

2. $Eff(\hat{\lambda}_{1\,MERSS},\hat{\lambda}_{1\,SRS})$ is fixed in ρ and increasing in the set size *m*.

3. $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS})$ changes over ρ and over the set size *m*.

4. $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS})$ is increasing in the set size *m* for fixed ρ.

5. $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS})$ is increasing in ρ for fixed set size *m*.

6. For any set size *m*, $Eff(\hat{\lambda}_{2\,MERSS},\hat{\lambda}_{2\,SRS})$ gets close to $Eff(\hat{\lambda}_{1\,MERSS},\hat{\lambda}_{1\,SRS})$ as ρ gets close to 1.

7. For example, for known $\rho = 0.9$ if we take *m*=5, then $Eff(\hat{\lambda}_{1\,MERSS},\hat{\lambda}_{1\,SRS}) = 2.33$, *i.e.*, with 100 units using MERSS we do as well as we do with 233 units using SRS.

Diab (2006) (see also Al-Saleh and Diab 2009) obtained the following table which contains the efficiency of the estimators of λ_1 and λ_2 using ranked set sampling $(\hat{\lambda}_{1\,RSS},\hat{\lambda}_{2\,RSS})$ with respect to the corresponding estimators using SRS $(\hat{\lambda}_{1\,SRS},\hat{\lambda}_{2\,SRS})$:

Table (2.2): The efficiency of $\hat{\lambda}_{1\,RSS}$ *w.r.t.* $\hat{\lambda}_{1\,SRS}$ ($\hat{\lambda}_{2\,RSS}$ *w.r.t.* $\hat{\lambda}_{2\,SRS}$)

ρ \ m	2	3	4	5
0.1	1.333(1.003)	1.636(1.004)	1.920(1.005)	2.190(1.005)
0.2	1.333(1.010)	1.636(1.016)	1.920(1.020)	2.190(1.022)
0.3	1.333(1.023)	1.636(1.036)	1.920(1.045)	2.190(1.051)
0.4	1.333(1.042)	1.636(1.066)	1.920(1.083)	2.190(1.095)
0.5	1.333(1.067)	1.636(1.108)	1.920(1.136)	2.190(1.157)
0.6	1.333(1.099)	1.636(1.163)	1.920(1.208)	2.190(1.243)
0.7	1.333(1.140)	1.636(1.235)	1.920(1.307)	2.190(1.363)
0.8	1.333(1.190)	1.636(1.331)	1.920(1.442)	2.190(1.533)
0.9	1.333(1.245)	1.636(1.460)	1.920(1.634)	2.190(1.786)

Comparing Table (2.1) with Table (2.2) we notice that the efficiency of $\hat{\lambda}_{1\,MERSS}$ w.r.t. $\hat{\lambda}_{1\,SRS}$ is greater than the efficiency of $\hat{\lambda}_{1\,RSS}$ w.r.t. $\hat{\lambda}_{1\,SRS}$ for all values of m and ρ, also the efficiency of $\hat{\lambda}_{2\,MERSS}$ w.r.t. $\hat{\lambda}_{2\,SRS}$ is greater than the efficiency of $\hat{\lambda}_{2\,RSS}$ w.r.t. $\hat{\lambda}_{2\,SRS}$ for all values of m and ρ, *i.e.,* we need less units using MERSS to do as well as RSS.

Consequently, we can easily obtain the following table which contains the efficiency of $\hat{\lambda}_{1\,MERSS}$ and $\hat{\lambda}_{2\,MERSS}$ with respect to the corresponding estimators using RSS.

Table (2.3): The efficiency of $\hat{\lambda}_{1\,MERSS}$ w.r.t $\hat{\lambda}_{1\,RSS}$ ($\hat{\lambda}_{2\,MERSS}$ w.r.t $\hat{\lambda}_{2\,RSS}$)

ρ \ m	2	3	4	5	6	10
0.1	1.042(1.002)	1.059(1.004)	1.065(1.007)	1.064(1.010)	1.059(1.012)	1.024(1.022)
0.2	1.042(1.006)	1.059(1.015)	1.065(1.024)	1.064(1.032)	1.059(1.041)	1.024(1.069)
0.3	1.042(1.012)	1.059(1.029)	1.065(1.046)	1.064(1.062)	1.059(1.077)	1.024(1.128)
0.4	1.042(1.020)	1.059(1.044)	1.065(1.069)	1.064(1.093)	1.059(1.115)	1.024(1.190)
0.5	1.042(1.026)	1.059(1.059)	1.065(1.092)	1.064(1.123)	1.059(1.151)	1.024(1.246)
0.6	1.042(1.033)	1.059(1.072)	1.065(1.111)	1.064(1.146)	1.059(1.180)	1.024(1.291)
0.7	1.042(1.039)	1.059(1.082)	1.065(1.122)	1.064(1.160)	1.059(1.195)	1.024(1.311)
0.8	1.042(1.043)	1.059(1.085)	1.065(1.123)	1.064(1.157)	1.059(1.189)	1.024(1.294)
0.9	1.042(1.044)	1.059(1.078)	1.065(1.106)	1.064(1.129)	1.059(1.150)	1.024(1.212)
0.99	1.042(1.042)	1.059(1.062)	1.065(1.070)	1.064(1.073)	1.059(1.071)	1.024(1.05)

Based on the above table we conclude the following:

1. $Eff(\hat{\lambda}_{1\,MERSS}, \hat{\lambda}_{1\,RSS})$ and $Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,RSS})$ are always greater than 1.
2. $Eff(\hat{\lambda}_{1\,MERSS}, \hat{\lambda}_{1\,RSS})$ is fixed in ρ and increasing in the set size $m \le 4$.
3. $Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,RSS})$ changes over ρ and over the set size m.
4. $Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,SRS})$ is increasing in the set size m for fixed ρ.

5. $Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,SRS})$ is increasing in $\rho \le 0.7$ for fixed set size m.

6. For any set size m, $Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,SRS})$ is increasing and equal the value of $Eff(\hat{\lambda}_{1\,MERSS}, \hat{\lambda}_{1\,SRS})$ for specific value of ρ, say ρ^* ($\rho^* < 1$), then it continue increasing until some value of ρ, say ρ^{**} ($\rho^{**} > \rho^*$), finally it is decreasing and equal the value of $Eff(\hat{\lambda}_{1\,MERSS}, \hat{\lambda}_{1\,SRS})$ again at $\rho = 1$.

The previous theoretical results are summarized in the following theorem:

Theorem (2-3)

a. When ρ is known, then:

$$\hat{\lambda}_{2\,MERSS} = \frac{1}{m(1-\rho) + \rho \sum_{k=1}^{m} \frac{(m+1)-k}{k}} \sum_{k=1}^{m} Y_{[k:k]}$$ is an unbiased estimator of λ_2.

b. $$Var(\hat{\lambda}_{2\,MERSS}) = \frac{\lambda_2^2}{\left(m(1-\rho) + \rho \sum_{k=1}^{m} \frac{(m+1)-k}{k}\right)^2} \left(m(1-\rho)^2 + 2\rho(1-\rho) \sum_{k=1}^{m} \frac{(m+1)-k}{k} + \rho^2 \sum_{k=1}^{m} \frac{(m+1)-k}{k^2} \right)$$

c. $$Eff(\hat{\lambda}_{2\,MERSS}, \hat{\lambda}_{2\,SRS}) = \frac{\left(m(1-\rho) + \rho \sum_{k=1}^{m} \frac{(m+1)-k}{k}\right)^2}{m\left(m(1-\rho)^2 + 2\rho(1-\rho) \sum_{k=1}^{m} \frac{(m+1)-k}{k} + \rho^2 \sum_{k=1}^{m} \frac{(m+1)-k}{k^2} \right)} \ge 1.$$

4. Best Linear Unbiased Estimator

In this section, we investigate the best linear unbiased estimator of λ_1 and λ_2 when ρ is known using MERSS and SRS, and find the variances of the two estimators. Also, we compare the variances of these estimators.

Assume that $\tau = \{\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_N\}$ is a set of unbiased estimators of a parameter θ. If $Var(\hat{\theta}_i) = \sigma_i^2$ and $Cov(\hat{\theta}_i, \hat{\theta}_j) = 0 \ \forall \ i \neq j$, then:

$$\hat{\theta}^* = \frac{\sum_{i=1}^{N} \hat{\theta}_i / \sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}$$

has the smallest variance; $Var(\hat{\theta}^*) = \frac{1}{\sum_{i=1}^{N} 1/\sigma_i^2}$, among all unbiased linear estimators ($\sum_{i=1}^{N} b_i \hat{\theta}_i$, b_i's are constants), therefore it is the BLUE of θ with $E_\theta\left(\sum_{i=1}^{N} b_i \hat{\theta}_i\right) = \theta$. **(See Casella and Berger (2002) page 363, Diab (2006) and Al-Saleh and Diab (2009))**

4.1. Best Linear Unbiased Estimator for λ_1

For simplicity, let

$$a_k = \sum_{j=1}^{k} \frac{1}{k-j+1} \quad \text{and} \quad b_k = \sum_{j=1}^{k} \frac{1}{(k-j+1)^2}.$$

We know from *Theorem (1-2)* that

a. $E(X_{(k:k)}) = a_k \lambda_1$.

and

b. $Var(X_{(k:k)}) = b_k \lambda_1^2$.

Therefore,

$$T_k = \frac{X_{(k:k)}}{a_k} \text{ is an unbiased estimator of } \lambda_1. \tag{17}$$

It follows that

$$Var(T_k) = Var\left(\frac{X_{(k:k)}}{a_k}\right) = \frac{1}{a_k^2} Var(X_{(k:k)}) = \frac{\lambda_1^2 b_k}{a_k^2}. \tag{18}$$

Thus, the best linear unbiased estimator of λ_1 is

$$T^* = \frac{\sum_{k=1}^{m} \frac{X_{(k:k)}}{a_k} \frac{a_k^2}{\lambda_1^2 b_k}}{\sum_{k=1}^{m} \frac{a_k^2}{\lambda_1^2 b_k}} = \frac{\sum_{k=1}^{m} \frac{a_k X_{(k:k)}}{b_k}}{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}. \tag{19}$$

It follows that

$$Var(T^*) = \frac{\lambda_1^2}{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}. \tag{20}$$

Hence, using (11) and (20) we get

$$Eff(T^*, \hat{\lambda}_{1\,SRS}) = \frac{Var(\hat{\lambda}_{1\,SRS})}{Var(T^*)} = \frac{\lambda_1^2}{m} \times \frac{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}{\lambda_1^2} = \frac{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}{m} = \frac{1}{m} \sum_{k=1}^{m} \left[\frac{\left(\sum_{j=1}^{k} \frac{1}{(k-j+1)} \right)^2}{\left(\sum_{j=1}^{k} \frac{1}{(k-j+1)^2} \right)} \right] \tag{21}$$

The following table gives the efficiency of $T^*\, w.r.t. \hat{\lambda}_{1\,SRS}$ ($\hat{\lambda}_{1\,MERSS}\, w.r.t. \hat{\lambda}_{1\,SRS}$)

Table (2.4): The efficiency of $T^*\, w.r.t. \hat{\lambda}_{1\,SRS}$ ($\hat{\lambda}_{1\,MERSS}\, w.r.t. \hat{\lambda}_{1\,SRS}$)

m	$Eff(T^*, \hat{\lambda}_{1\,SRS})$	$Eff(\hat{\lambda}_{1\,MERSS}\, w.r.t\, \hat{\lambda}_{1\,SRS})$
2	1.400	1.389
3	1.756	1.733
4	2.080	2.044
5	2.376	2.330

Based on Table (2.4) we conclude the following:

1. $Eff(T^*, \hat{\lambda}_{1\,SRS})$ is always greater than 1.

2. $Eff(T^*, \hat{\lambda}_{1\,SRS})$ is larger than $Eff(\hat{\lambda}_{1\,MERSS}\ w.r.t\ \hat{\lambda}_{1\,SRS})$ for all of the set size m, i.e, T^* is more significantly efficient than $\hat{\lambda}_{1\,MERSS}$ and hence than $\hat{\lambda}_{1\,RSS}$ and $\hat{\lambda}_{1\,SRS}$.

Notice

$$T^* = \sum_{k=1}^{m} u_{kk} \frac{X_{(k:k)}}{a_k} \text{ where } u_{kk} = \frac{\frac{a_k^2}{b_k}}{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}.$$

Also,

$$\hat{\lambda}_{1\,MERSS} = \sum_{k=1}^{m} v_{kk} \frac{X_{(k:k)}}{a_k} \text{ where } v_{kk} = \frac{a_k}{\sum_{k=1}^{m} a_k}.$$

So,

$$\frac{v_{kk}}{u_{kk}} = \frac{a_k}{\sum_{k=1}^{m} a_k} \times \frac{\sum_{k=1}^{m} \frac{a_k^2}{b_k}}{\frac{a_k^2}{b_k}},$$

$$\frac{v_{11}}{u_{11}} = 1, \frac{v_{22}}{u_{22}} = 0.933, \frac{v_{33}}{u_{33}} = 0.903, \frac{v_{44}}{u_{44}} = 0.886, \frac{v_{55}}{u_{55}} = 0.875$$

Hence, the best linear unbiased estimator and the traditional one are different but relatively close to each other.

The previous theoretical results are summarized in the following theorem:

Theorem (2-4)

a. $$T^* = \sum_{k=1}^{m} \left[\frac{\sum_{j=1}^{k} \frac{1}{k-j+1} X_{(k:k)}}{\sum_{j=1}^{k} \frac{1}{(k-j+1)^2}} \right] \Bigg/ \sum_{k=1}^{m} \left[\frac{\left(\sum_{j=1}^{k} \frac{1}{k-j+1} \right)^2}{\sum_{j=1}^{k} \frac{1}{(k-j+1)^2}} \right]$$ is the BLUE of λ_1.

b. $$Var(T^*) = \lambda_1^2 \left(\sum_{k=1}^{m} \left[\frac{\left(\sum_{j=1}^{k} \frac{1}{k-j+1} \right)^2}{\sum_{j=1}^{k} \frac{1}{(k-j+1)^2}} \right] \right)^{-1}.$$

c. $$Eff(T^*, \hat{\lambda}_{1\,SRS}) = \frac{1}{m}\sum_{k=1}^{m}\left[\frac{\left(\sum_{j=1}^{k}\frac{1}{(k-j+1)}\right)^2}{\left(\sum_{j=1}^{k}\frac{1}{(k-j+1)^2}\right)}\right].$$

4.2. Best Linear Unbiased Estimator for λ_2

We know from *Theorem (2-2)* that

a. $E(Y_{[k:k]}) = (1-\rho)\lambda_2 + \rho a_k \lambda_2$,

and

b. $Var(Y_{[k:k]}) = \lambda_2^2\left((1-\rho)^2 + 2\rho(1-\rho)a_k + b_k\rho^2\right)$.

Therefore,

$$\omega_k = \frac{Y_{[k:k]}}{(1-\rho)+\rho a_k} \text{ is an unbiased estimator of } \lambda_2. \qquad (22)$$

It follows that

$$Var(\omega_k) = Var\left(\frac{Y_{[k:k]}}{1-\rho+\rho a_k}\right) = \frac{1}{(1-\rho+\rho a_k)^2}Var(Y_{[k:k]}).$$

Using (8) in the previous equation we get

$$Var(\omega_k) = \lambda_2^2\frac{(1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k}{(1-\rho+\rho a_k)^2}. \qquad (23)$$

Thus, the best linear unbiased estimator of λ_2 is

$$\omega^* = \sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)Y_{[k:k]}}{\lambda_1^2\left[(1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right]}\Bigg/\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\lambda_1^2\left[(1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right]}. \qquad (24)$$

It follows that

$$Var(\omega^*) = Var\left(\sum_{k=1}^{m}\frac{Y_{[k:k]}}{1-\rho+\rho a_k}\frac{(1-\rho+\rho a_k)^2}{\lambda_1^2\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right)}\Bigg/\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\lambda_1^2\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right.}\right.$$

$$= \sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2 Var(Y_{[k:k]})}{\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right)^2}\Bigg/\left(\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right)}\right)^2.$$

Applying (8), we get

$$Var(\omega^*) = \lambda_2^2\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right)}\Bigg/\left(\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\left((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k\right)}\right)^2 \qquad (25)$$

Hence, using (15) and (25) we get

$$Eff(\omega^*,\hat{\lambda}_{2\,SRS}) = \frac{Var(\hat{\lambda}_{2\,SRS})}{Var(\omega^*)} = \left(\sum_{k=1}^{m} \frac{((1-\rho)+\rho a_k)^2}{((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k)}\right)^2 \Big/ m\sum_{k=1}^{m} \frac{((1-\rho)+\rho a_k)^2}{((1-\rho)^2 + 2\rho(1-\rho)a_k + \rho^2 b_k)} \quad (26)$$

Table (2.5) gives the efficiency of ω^* w.r.t. $\hat{\lambda}_{2\,SRS}$ ($\hat{\lambda}_{2\,MERSS}$ w.r.t. $\hat{\lambda}_{2\,SRS}$)

Table (2.5): The efficiency of ω^* w.r.t. $\hat{\lambda}_{2\,SRS}$ ($\hat{\lambda}_{2\,MERSS}$ w.r.t. $\hat{\lambda}_{2\,SRS}$)

ρ \ m	2	3	4	5
0.1	1.005(1.004)	1.009(1.008)	1.013(1.012)	1.016(1.015)
0.2	1.017(1.016)	1.032(1.031)	1.046(1.044)	1.057(1.055)
0.3	1.037(1.035)	1.068(1.066)	1.095(1.093)	1.119(1.116)
0.4	1.063(1.061)	1.115(1.113)	1.160(1.158)	1.199(1.197)
0.5	1.095(1.095)	1.174(1.173)	1.241(1.241)	1.300(1.299)
0.6	1.135(1.135)	1.247(1.247)	1.342(1.342)	1.425(1.425)
0.7	1.184(1.184)	1.337(1.336)	1.467(1.467)	1.582(1.581)
0.8	1.242(1.241)	1.446(1.444)	1.623(1.619)	1.780(1.774)
0.9	1.313(1.309)	1.583(1.574)	1.821(1.808)	2.035(2.017)

Based on the previous table we conclude the following:

1. $Eff(\omega^*,\hat{\lambda}_{2\,SRS})$ is always greater than 1.
2. $Eff(\omega^*,\hat{\lambda}_{2\,SRS})$ is increasing in the set size m for fixed ρ, and also increasing in ρ for fixed set size m.
3. $Eff(\omega^*,\hat{\lambda}_{2\,SRS})$ is larger than $Eff(\hat{\lambda}_{2\,MERSS}$ w.r.t $\hat{\lambda}_{2\,SRS})$ for any set size m and ρ, i.e., ω^* is more efficient than $\hat{\lambda}_{2\,MERSS}$ and hence than $\hat{\lambda}_{2\,RSS}$ and $\hat{\lambda}_{2\,SRS}$.

Notice

$$\omega^{*} = \sum_{k=1}^{m} Y_{[k:k]} \left[\frac{1-\rho+\rho a_k}{(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k}\right] \Bigg/ \sum_{k=1}^{m}\left[\frac{(1-\rho+\rho a_k)^2}{(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k}\right]$$

$$\omega^{*} = \sum_{k=1}^{m} c_{km} Y_{[k:k]}\text{, where}$$

$$c_{km} = \frac{(1-\rho+\rho a_k)}{[(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k]} \Bigg/ \sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{[(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k]},$$

$$\hat{\lambda}_{2\,MERSS} = \frac{1}{m(1-\rho)+\rho S_m}\sum_{k=1}^{m} d_{km} Y_{[k:k]}\,.$$

The previous theoretical results are summarized in the following Theorem:

Theorem (2-5)

a. $$\omega^{*} = \frac{(1-\rho+\rho a_k)Y_{[k:k]}}{\lambda_1^2[(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k]} \Bigg/ \sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{\lambda_1^2[(1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k]}$$

is the BLUE of λ_2.

b. $$Var(\omega^{*}) = \lambda_2^2 \sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{((1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 b_k)} \Bigg/ \left(\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{((1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 v_k)}\right)^2$$

c. $$Eff(\omega^{*},\hat{\lambda}_{2\,SRS}) = \frac{Var(\hat{\lambda}_{2\,SRS})}{Var(\omega^{*})} = \left(\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{((1-\rho)^2+2\rho(1-\rho)a_k+\rho^2 v_k)}\right)^2 \Bigg/ m\sum_{k=1}^{m}\frac{(1-\rho+\rho a_k)^2}{((1-\rho)^2+2\rho(1-\rho)a_k+\rho^2}$$

5. Estimation of the Correlation Coefficient Using MERSS When λ_1 and λ_2 are Known

In this section, we consider the estimation of ρ when λ_1 and λ_2 are known using MERSS and SRS, and find the variances of the two estimators. Also, we compare the variances of these estimators.

Diab (2006), (see also Al-Saleh and Diab (2009)), derived the following formula for the variance of $\hat{\rho}$ based on SRS:

$$Var(\hat{\rho}_{1SRS}) = \frac{3\rho^2 + 14\rho + 3}{m}, \text{ where } \hat{\rho}_{1SRS} = \frac{\sum_{i=1}^{m} X_i Y_i}{m\lambda_1\lambda_2} - 1 \quad (27)$$

Now, suppose that λ_1 and λ_2 are known, then,

$$E(X_{(k:k)}Y_{[k:k]}) = E(E(X_{(k:k)}Y_{[k:k]} \mid X_{(k:k)})) = E(X_{(k:k)}E(Y_{[k:k]} \mid X_{(k:k)}))$$

$$= E\left[X_{(k:k)}\left((1-\rho)\lambda_2 + \rho\frac{\lambda_2}{\lambda_1}X_{(k:k)}\right)\right]$$

$$= E\left((1-\rho)\lambda_2 X_{(k:k)} + \rho\frac{\lambda_2}{\lambda_1}X^2_{(k:k)}\right)$$

$$= (1-\rho)\lambda_2 E(X_{(k:k)}) + \rho\frac{\lambda_2}{\lambda_1}E(X^2_{(k:k)})$$

$$= (1-\rho)\lambda_2\lambda_1\sum_{j=1}^{k}\frac{1}{k-j+1} + \rho\frac{\lambda_2}{\lambda_1}E(X^2_{(k:k)}) .$$

But,

$$E(X^2_{(k:k)}) = Var(X_{(k:k)}) + [E(X_{(k:k)})]^2$$

$$= \lambda_1^2\sum_{j=1}^{k}\frac{1}{(k-j+1)^2} + \lambda_1^2\left[\sum_{j=1}^{k}\frac{1}{k-j+1}\right]^2 .$$

Then,

$$E(X_{(k:k)}Y_{[k:k]}) = (1-\rho)\lambda_2\lambda_1\sum_{j=1}^{k}\frac{1}{k-j+1} + \rho\frac{\lambda_2}{\lambda_1}\left(\lambda_1^2\sum_{j=1}^{k}\frac{1}{(k-j+1)^2} + \lambda_1^2\left[\sum_{j=1}^{k}\frac{1}{k-j+1}\right]^2\right)$$

$$= \lambda_1\lambda_2 \sum_{j=1}^{k} \frac{1}{k-j+1} + \rho\left(\lambda_1\lambda_2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2} - \lambda_1\lambda_2 \sum_{j=1}^{k} \frac{1}{k-j+1} + \lambda_1\lambda_2 \left[\sum_{j=1}^{k} \frac{1}{k-j+1} \right]^2 \right).$$

Hence,

$$E\left(\sum_{k=1}^{m} X_{(k:k)}Y_{[k:k]} \right) = \rho\left(\lambda_1\lambda_2 C_m - \lambda_1\lambda_2 S_m + \lambda_1\lambda_2 \sum_{k=1}^{m} \left[\sum_{j=1}^{k} \frac{1}{k-j+1} \right]^2 \right) + \lambda_1\lambda_2 S_m. \tag{28}$$

Thus,

$$\frac{E\left(\sum_{k=1}^{m} X_{(k:k)}Y_{[k:k]} \right) - \lambda_1\lambda_2 S_m}{\lambda_1\lambda_2 C_m - \lambda_1\lambda_2 S_m + \lambda_1\lambda_2 \sum_{k=1}^{m} \left[\sum_{j=1}^{k} \frac{1}{k-j+1} \right]^2} = \rho. \tag{29}$$

For simplicity, let

$$a_k = \sum_{j=1}^{k} \frac{1}{k-j+1} \text{ and } b_k = \sum_{j=1}^{k} \frac{1}{(k-j+1)^2}.$$

Hence,

$$\hat{\rho}_{1MERSS} = \frac{\dfrac{\sum_{k=1}^{m} X_{(k:k)}Y_{[k:k]}}{\lambda_1\lambda_2} - S_m}{C_m - S_m + \sum_{k=1}^{m} \left[\sum_{j=1}^{k} \frac{1}{k-j+1} \right]^2} = \frac{\dfrac{\sum_{k=1}^{m} X_{(k:k)}Y_{[k:k]}}{\lambda_1\lambda_2} - S_m}{C_m - S_m + \sum_{k=1}^{m} a_k^2} \tag{30}$$

is an unbiased estimator of ρ.

Notice that $\hat{\rho}_{1MERSS}$ is the naive (unmodified) estimator.

It follows that

$$Var(\hat{\rho}_{1MERSS}) = \frac{\sum_{k=1}^{m} Var\left(X_{(k:k)}Y_{[k:k]} \right)}{\lambda_1^2\lambda_2^2 \left(C_m - S_m + \sum_{k=1}^{m} a_k^2 \right)^2}. \tag{31}$$

It is clear that,

$$Var(X_{(k:k)}Y_{[k:k]}) = E(Var(X_{(k:k)}Y_{[k:k]} \mid X_{(k:k)})) + Var(E(X_{(k:k)}Y_{[k:k]} \mid X_{(k:k)}))$$

$$= E(X_{(k:k)}^2 Var(Y_{[k:k]} \mid X_{(k:k)})) + Var(X_{(k:k)}E(Y_{[k:k]} \mid X_{(k:k)})).$$

But,

$$E\left(X_{(k:k)}^2 Var\left(Y_{[k:k]} \mid X_{(k:k)}\right)\right) = E\left(X_{(k:k)}^2 Var(Y \mid X)\right)$$

$$= E\left((1-\rho)^2 \lambda_2^2 X_{(k:k)}^2 + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1} X_{(k:k)}^3\right)$$

$$= (1-\rho)^2 \lambda_2^2 E(X_{(k:k)}^2) + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1} E(X_{(k:k)}^3) \qquad (32)$$

Furthermore,

$$E(X_{(k:k)}^2) = \lambda_1^2 \sum_{j=1}^{k} \frac{1}{(k-j+1)^2} + \lambda_1^2 \left[\sum_{j=1}^{k} \frac{1}{k-j+1}\right]^2$$

$$= \lambda_1^2 b_k + \lambda_1^2 a_k^2 \qquad (33)$$

Also,

$$E(X_{(k:k)}^3) = \frac{k}{\lambda_1} \int_0^\infty x^3 \left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right)^{k-1} \exp\left(-\frac{x}{\lambda_1}\right) dx.$$

But,

$$\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} \exp\left(\frac{-jx}{\lambda_1} - \frac{x}{\lambda_1}\right) \exp\left(\frac{x}{\lambda_1}\right)(-1)^j,$$

Therefore,

$$E(X_{(k:k)}^3) = \frac{k}{\lambda_1} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \int_0^\infty x^3 \exp\left(-\frac{(j+1)x}{\lambda_1}\right) dx,$$

But,

$$\int_0^\infty x^3 \exp\left(-\frac{(j+1)x}{\lambda_1}\right) dx = \frac{6\lambda_1^4}{(1+j)^4},$$

Thus,

$$E(X_{(k:k)}^3) = \frac{k}{\lambda_1} \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{6\lambda_1^4}{(1+j)^4} = 6k\lambda_1^3 \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(1+j)^4}. \qquad (34)$$

Now, applying (33) and (34) in (32) we get

$$E(X_{(k:k)}^2 Var(Y_{[k:k]} \mid X_{(k:k)})) =$$

$$(1-\rho)^2 \lambda_1^2 \lambda_2^2 \left\{\sum_{j=1}^{k} \frac{1}{(k-j+1)^2} + \left[\sum_{j=1}^{k} \frac{1}{k-j+1}\right]^2\right\} + 2\rho(1-\rho)\frac{\lambda_2^2}{\lambda_1}\left\{6k\lambda_1^3 \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(1+j)^4}\right\}$$

$$= \lambda_1^2 \lambda_2^2 \left[(1-\rho)^2 b_k + (1-\rho)^2 (a_k)^2 + 12k\rho(1-\rho) \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(1+j)^4}\right] \qquad (35)$$

Now,

$$Var(X_{(k:k)}E(Y_{[k:k]}\mid X_{(k:k)}))=Var\left(X_{(k:k)}(1-\rho)\lambda_2+\rho\frac{\lambda_2}{\lambda_1}X_{(k:k)}^2\right)$$

$$=E\left(X_{(k:k)}(1-\rho)\lambda_2+\rho\frac{\lambda_2}{\lambda_1}X_{(k:k)}^2\right)^2-\left[E\left(X_{(k:k)}(1-\rho)\lambda_2+\rho\frac{\lambda_2}{\lambda_1}X_{(k:k)}^2\right)\right]^2$$

$$=E\left(X_{(k:k)}^2(1-\rho)^2\lambda_2^2+2X_{(k:k)}^3(1-\rho)\rho\frac{\lambda_2^2}{\lambda_1}+\rho^2\frac{\lambda_2^2}{\lambda_1^2}X_{(k:k)}^4\right)-\left[(1-\rho)\lambda_2E(X_{(k:k)})+\rho\frac{\lambda_2}{\lambda_1}E(X_{(k:k)}^2)\right]^2$$

$$=(1-\rho)^2\lambda_2^2E(X_{(k:k)}^2)+2(1-\rho)\rho\frac{\lambda_2^2}{\lambda_1}E(X_{(k:k)}^3)+\rho^2\frac{\lambda_2^2}{\lambda_1^2}E(X_{(k:k)}^4)-\left[(1-\rho)\lambda_2E(X_{(k:k)})+\rho\frac{\lambda_2}{\lambda_1}E(X_{(k:k)}^2)\right]^2 \quad (36)$$

Also,

$$E(X_{(k:k)}^4)=\frac{k}{\lambda_1}\int_0^\infty x^4\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right)^{k-1}\exp\left(-\frac{x}{\lambda_1}\right)dx$$

$$=\frac{k}{\lambda_1}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\int_0^\infty x^4\exp\left(-\frac{(j+1)x}{\lambda_1}\right)dx.$$

But,

$$\int_0^\infty x^4\exp\left(-\frac{(j+1)x}{\lambda_1}\right)dx=\frac{24\lambda_1^5}{(1+j)^5},$$

Thus,

$$E(X_{(k:k)}^4)=\frac{k}{\lambda_1}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\frac{24\lambda_1^5}{(1+j)^5}=24k\lambda_1^4\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5} \quad (37)$$

Now using (5), (33), (34), (37) in (36) we get

$$Var(X_{(k:k)}E(Y_{[k:k]}\mid X_{(k:k)}))=$$

$$\Bigg\{(1-\rho)^2\lambda_1^2\lambda_2^2\left(b_k+a_k^2\right)+2(1-\rho)\rho\frac{\lambda_2^2}{\lambda_1}\left(6k\lambda_1^3\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+$$

$$\rho^2\frac{\lambda_2^2}{\lambda_1^2}\left(24k\lambda_1^4\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\left[(1-\rho)\lambda_2\left(\lambda_1a_k\right)+\rho\frac{\lambda_2}{\lambda_1}\left(\lambda_1^2b_k+\lambda_1^2a_k^2\right)\right]^2\Bigg\} \quad (38)$$

Also, using (37) and (38) we get

$$Var\left(X_{(k:k)}Y_{[k:k]}\right)=$$

$$\lambda_1^2\lambda_2^2\left\{(1-\rho)^2\left(b_k+a_k^2\right)+2(1-\rho)\rho\left(6k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+\right.$$

$$\left.\rho^2\left(24k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\left[(1-\rho)(a_k)+\rho\left(b_k+a_k^2\right)\right]^2\right\}$$

$$Var\left(X_{(k:k)}Y_{[k:k]}\right)=$$

$$\lambda_1^2\lambda_2^2\left\{2(1-\rho)^2\left(b_k+a_k^2\right)+24(1-\rho)\rho\left(k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+\rho^2\left(24k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\left[(1-\rho)(a_k)+\rho\left(b_k+a_k^2\right)\right.\right.$$

$$Var\left(\sum_{k=1}^{m}X_{(k:k)}Y_{[k:k]}\right)=$$

$$=\lambda_1^2\lambda_2^2\left\{2(1-\rho)^2\sum_{k=1}^{m}\left(b_k+a_k^2\right)+24(1-\rho)\rho\left(\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+\rho^2\left(24\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\sum_{k=1}^{m}\left[(1-\rho)(a_k)+\rho\left(b_k+a_k^2\right)\right]^2\right\}$$

Thus,

$$Var(\hat{\rho}_{1MERSS})=$$

$$\frac{\left\{2(1-\rho)^2\sum_{k=1}^{m}\left(b_k+a_k^2\right)+24(1-\rho)\rho\left(\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+\rho^2\left(24\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\sum_{k=1}^{m}\left[(1-\rho)(a_k)+\rho\left(b_k+a_k^2\right)\right]^2\right\}}{\left(C_m-S_m+\sum_{k=1}^{m}a_k^2\right)^2}. \quad (39$$

Hence, using (27) and (39) we get

$$Eff(\hat{\rho}_{1MERSS},\hat{\rho}_{1SRS})=\frac{Var(\hat{\rho}_{1SRS})}{Var(\hat{\rho}_{1MERSS})}$$

$$\frac{(3\rho^2+14\rho+3)\left(C_m-S_m+\sum_{k=1}^{m}a_k^2\right)^2}{m\left\{2(1-\rho)^2\sum_{k=1}^{m}\left(b_k+a_k^2\right)+24(1-\rho)\rho\left(\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right)+\rho^2\left(24\sum_{k=1}^{m}k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right)-\sum_{k=1}^{m}\left[(1-\rho)(a_k)+\rho\left(b_k+a_k^2\right)\right]^2\right\}}.($$

The previous theoretical results are summarized in the following theorem:

Theorem (2-6)

a. $$\hat{\rho}_{1MERSS} = \frac{\dfrac{\sum_{k=1}^{m} X_{(k:k)}Y_{[k:k]}}{\lambda_1\lambda_2} - S_m}{\left(C_m - S_m + \sum_{k=1}^{m} a_k^2\right)}$$ is an unbiased estimator of ρ.

b. $$Var(\hat{\rho}_{1MERSS}) = \frac{\sum_{k=1}^{m} Var\left(X_{(k:k)}Y_{[k:k]}\right)}{\lambda_1^2\lambda_2^2\left(C_m - S_m + \sum_{k=1}^{m} a_k^2\right)^2}$$

c. $$Eff(\hat{\rho}_{1MERSS}, \hat{\rho}_{1SRS}) = \frac{Var(\hat{\rho}_{1SRS})}{Var(\hat{\rho}_{1MERSS})}$$

$$= \frac{(3\rho^2 + 14\rho + 3)\left(C_m - S_m + \sum_{k=1}^{m} a_k^2\right)^2}{m\left\{2(1-\rho)^2\sum_{k=1}^{m}\left(b_k + a_k^2\right) + 24(1-\rho)\rho\left(\sum_{k=1}^{m} k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^4}\right) + \rho^2\left(24\sum_{k=1}^{m} k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(1+j)^5}\right) - \sum_{k=1}^{m}\left[(1-\rho)a_k + \rho\left(b_k + a_k^2\right)\right]^2\right\}}$$

Table (2.6) contains the efficiency of $\hat{\rho}_{1MERSS}$ $w.r.t$ $\hat{\rho}_{1SRS}$.

Table (2.6): The efficiency of $\hat{\rho}_{1MERSS}$ $w.r.t$ $\hat{\rho}_{1SRS}$

ρ \ m	**2**	**3**	**4**	**5**
0.1	1.700	2.404	3.091	3.752
0.2	1.681	2.359	3.014	3.644
0.3	1.670	2.335	2.977	3.591
0.4	1.664	2.322	2.958	3.565
0.5	1.661	2.316	2.948	3.555
0.6	1.659	2.312	2.945	3.553
0.7	1.658	2.312	2.946	3.556
0.8	1.658	2.313	2.950	3.564
0.9	1.658	2.315	2.955	3.573
0.99	1.659	2.317	2.961	3.583

Based on Table (2.6) we conclude the following:

1- The efficiency of $\hat{\rho}_{1MERSS}$ $w.r.t.$ $\hat{\rho}_{1SRS}$ is always larger than 1.

2- The efficiency of $\hat{\rho}_{1MERSS}$ $w.r.t.$ $\hat{\rho}_{1SRS}$ is increasing in the set size m for fixed ρ.

3- The efficiency of $\hat{\rho}_{1MERSS}$ $w.r.t.$ $\hat{\rho}_{1SRS}$ is decreasing in ρ for fixed set size m.

6. Estimation of the Correlation Coefficient Using MERSS When λ_1 and λ_2 are Unknown

Suppose that λ_1 and λ_2 are unknown, then we can replace them in $\hat{\rho}_{1SRS}$ by $\hat{\lambda}_{1SRS}$ and $\hat{\lambda}_{2SRS}$ to get

$$\hat{\rho}_{2SRS} = \frac{\sum_{i=1}^{m} X_i Y_i}{m \hat{\lambda}_{1SRS} \hat{\lambda}_{2SRS}} - 1 \tag{41}$$

Since $0 \le \rho < 1$, we modify this estimator of ρ by:

$$\hat{\rho}^{\bullet}_{2\,SRS} = \begin{cases} 0 & if \quad \hat{\rho}_{2\,SRS} < 0 \\ \hat{\rho}_{2\,SRS} & if \quad 0 \le \hat{\rho}_{2\,SRS} \le 1 \\ 1 & if \quad \hat{\rho}_{2\,SRS} > 1 \end{cases}$$

Similarly,

$$\hat{\rho}_{2MERSS} = \frac{\dfrac{\sum_{k=1}^{m} X_{(k:k)} Y_{[k:k]}}{\hat{\lambda}_{1MERSS} \hat{\lambda}^{\bullet}_{2\,MERSS}} - S_m}{C_m - S_m + \sum_{k=1}^{m} a_k^2}, \tag{42}$$

where $\hat{\lambda}^{\bullet}_{2\,MERSS} = \dfrac{\sum_{k=1}^{m} Y_{[k:k]}}{m}$ since $\hat{\lambda}_{2MERSS}$ depends on ρ.

Since $0 \le \rho < 1$, we modify the estimator of ρ by:

$$\hat{\rho}^{\bullet}_{2\,MERSS} = \begin{cases} 0 & if \quad \hat{\rho}_{2\,MERSS} < 0 \\ \hat{\rho}_{2\,MERSS} & if \quad 0 \le \hat{\rho}_{2\,MERSS} \le 1 \\ 1 & if \quad \hat{\rho}_{2\,MERSS} > 1 \end{cases}$$

We compare the bias and MSE of the two estimators $\hat{\rho}^{\bullet}_{2\,MERSS}$ and $\hat{\rho}^{\bullet}_{2\,SRS}$ via simulation, using (10000) iterations.

To simulate (X,Y) from DBED $(\lambda_1,\lambda_2,\rho)$, **Iliopoulos (2003)** rewrote the pdf of Downton (1) as an infinite mixture of independent gamma distributions with geometric mixing distribution as **(See also Diab (2006) ,Al-Saleh and Diab 2009)**):

$$f(x,y;\lambda_1,\lambda_2,\rho) = \sum_{k=0}^{\infty} \pi(k;\rho)\Gamma_{(k+1)}(x;\lambda_1(1-\rho))\Gamma_{(k+1)}(y;\lambda_2(1-\rho)),$$

where $\Gamma_{\alpha}(.;\beta)$ is the pdf of gamma(α,β) and $\pi(k;\rho) = (1-\rho)\rho^k, k = 0,1,2,...$ is the geometric pmf. Let K be a random variable with this geometric pdf. Then X and Y are conditionally (given $K = k$) independent gamma random variables with $\alpha = k+1$ and $\beta_1 = \lambda_1(1-\rho)$, $\beta_2 = \lambda_2(1-\rho)$, respectively. Also, the unconditional distribution of (X,Y) is a DBED $(\lambda_1,\lambda_2,\rho)$.

The algorithm of simulating observations form the DBED can be done using the following steps:

1. Given λ_1,λ_2 and ρ; simulate m random numbers from the geometric distribution with $p = (1-\rho)$, say k_i, $i = 1,...,m$.
2. For each k_i, simulate two independent random variable X_i and Y_i, where X is from $\Gamma(k_i+1;\lambda_1(1-\rho))$, and Y_i is from $\Gamma(k_i+1;\lambda_2(1-\rho))$.
3. Use the data in (2) to compute the values of $\hat{\rho}^{\bullet}_{2\,SRS}$ and $\hat{\rho}^{\bullet}_{2\,MERSS}$.
4. Repeat Steps (1-3) 10000 times to get 10000 values of $\hat{\rho}^{\bullet}_{2\,SRS}$ and $\hat{\rho}^{\bullet}_{2\,MERSS}$.

Bias and MSE of each of the estimators ($\hat{\rho}^{\bullet}_{2\,SRS}, \hat{\rho}^{\bullet}_{2\,MERSS}$) are given in Tables (2.7), (2.8) respectively, and the efficiency of $\hat{\rho}^{\bullet}_{2\,MERSS}$ *w.r.t.* $\hat{\rho}^{\bullet}_{2\,SRS}$ is given in Table (2.9).

Table (2.7): The bias of $\hat{\rho}^{\bullet}_{2\,SRS}$ ($\hat{\rho}^{\bullet}_{2\,MERSS}$)

ρ \ m	2	3	4	5
0.1	0.0376439 (0.0107227)	0.0584117 (0.000724853)	0.0586485 (-0.0120098)	0.0548856 (-0.0204013)
0.2	-0.0523795 (-0.0793768)	-0.0233886 (-0.0801307)	-0.0184251 (-0.0916448)	-0.00529215 (-0.104122)
0.3	-0.135611 (-0.171955)	-0.0981019 (-0.163776)	-0.0844414 (-0.176105)	-0.075403 (-0.18657)
0.4	-0.225323 (-0.257008)	-0.176568 (-0.246904)	-0.149973 (-0.257058)	-0.126892 (-0.266765)
0.5	-0.310434 (-0.342278)	-0.239222 (-0.32875)	-0.205666 (-0.336086)	-0.186279 (-0.348883)
0.6	-0.390079 (-0.430483)	-0.312983 (-0.413311)	-0.270314 (-0.418829)	-0.240321 (-0.431649)
0.7	-0.468765 (-0.513701)	-0.378034 (-0.491661)	-0.329873 (-0.501695)	-0.290391 (-0.513443)
0.8	-0.538559 (-0.590251)	-0.439845 (-0.571455)	-0.375437 (-0.581406)	-0.330727 (-0.595339)
0.9	-0.61117 (-0.672461)	-0.493015 (-0.645387)	-0.416008 (-0.661558)	-0.373616 (-0.679775)

Table (2.8): The MSE of $\hat{\rho}^{*}_{2\,SRS}$ $(\hat{\rho}^{*}_{2\,MERSS})$

ρ \ m	2	3	4	5
0.1	0.0442721 (0.0293167)	0.0601832 (0.0238382)	0.0582355 (0.0183293)	0.0555794 (0.0140678)
0.2	0.0490005 (0.0370344)	0.0637464 (0.0348262)	0.0650449 (0.0301238)	0.067288 (0.0275962)
0.3	0.0707885 (0.0618572)	0.0809181 (0.0598328)	0.0807965 (0.0552902)	0.0801482 (0.0529677)
0.4	0.105702 (0.102499)	0.106769 (0.0962038)	0.105989 (0.0934151)	0.102262 (0.0925348)
0.5	0.154502 (0.156786)	0.142506 (0.147673)	0.132283 (0.142743)	0.127251 (0.144423)
0.6	0.214792 (0.227261)	0.187929 (0.210604)	0.167288 (0.205588)	0.15389 (0.20876)
0.7	0.288881 (0.308422)	0.235174 (0.284579)	0.204804 (0.283881)	0.179919 (0.287066)
0.8	0.364987 (0.397526)	0.289395 (0.371836)	0.238071 (0.370711)	0.204345 (0.378874)
0.9	0.453857 (0.504155)	0.340542 (0.463529)	0.26872 (0.47161)	0.230149 (0.485024)

Table (2.9): The efficiency of $\hat{\rho}^{*}_{2\,MERSS}$ w.r.t. $\hat{\rho}^{*}_{2\,SRS}$

ρ \ m	2	3	4	5
0.1	1.51013	2.52465	3.17719	3.95083
0.2	1.32311	1.83041	2.15925	2.43831
0.3	1.14438	1.3524	1.46132	1.51315
0.4	1.03124	1.10982	1.13461	1.10512
0.5	0.985437	0.965008	0.926721	0.881097
0.6	0.945132	0.892336	0.813708	0.737164
0.7	0.936641	0.826392	0.721445	0.626752
0.8	0.918146	0.778287	0.6422	0.539347
0.9	0.900233	0.734673	0.569792	0.474511

Based on Tables (2.7, 2.8 and 2.9) we conclude the following:

1- The absolute value of bias ($\hat{\rho}^{*}_{2\,SRS}$) is increasing in $m \geq 3$ for fixed ρ.

2- The value of MSE ($\hat{\rho}^{*}_{2\,SRS}$) is decreasing in the set size $m \leq 4$ for fixed ρ.

3- The absolute value of bias ($\hat{\rho}^{*}_{2\,MERSS}$) is decreasing in the set size m for $\rho \geq 0.2$.

4- The value of MSE ($\hat{\rho}^{*}_{2\,MERSS}$) is decreasing in the set size m for $\rho \geq 0.4$.

5- The efficiency of $\hat{\rho}^{*}_{2\,MERSS}$ $w.r.t$ $\hat{\rho}^{*}_{2\,SRS}$ is larger than 1 for $\rho \leq 0.4$.

7. Estimation of λ_1 and λ_2 Using MERSS When ρ is Unknown

We noticed that the formula of $\hat{\lambda}_{1\,MERSS} = \frac{1}{S_m}\sum_{k=1}^{m} X_{(k:k)}$ does not depend on ρ, while $\hat{\lambda}_{2\,MERSS}$ does. Suppose that ρ is unknown, then we can replace it by $\hat{\rho}^{*}_{2\,MERSS}$ in the formula of $\hat{\lambda}_{2\,MERSS}$ to get

$$\hat{\lambda}^{*}_{2\,MERSS} = \frac{1}{m(1-\hat{\rho}^{*}_{2\,MERSS}) + \hat{\rho}^{*}_{2\,MERSS}\, S_m}\sum_{k=1}^{m} Y_{[k:k]} \quad (43)$$

Bias and MSE of each of the estimators ($\hat{\lambda}_{2\,SRS}, \hat{\lambda}^{*}_{2\,MERSS}$) are given in Tables (2.10), (2.11) respectively, and the efficiency of $\hat{\lambda}^{*}_{2\,MERSS}$ $w.r.t.\ \hat{\lambda}_{2\,SRS}$ is given in Table (2.12).

Table (2.10): The bias of $\hat{\lambda}_{2\,SRS}(\hat{\lambda}^{*}_{2\,MERSS})$

ρ \ m	2	3	4	5
0.1	-0.00279574 (-0.0020933)	-0.00168242 (-0.000984182)	-0.00472256 (0.00819218)	-0.0040465 (0.0191077)
0.2	0.00016427 (0.0160948)	-0.00109245 (0.0359209)	0.00272019 (0.0563623)	0.000303165 (0.0808028)
0.3	-0.0000583597 (0.0370854)	-0.000877509 (0.0693029)	0.00441161 (0.0948296)	0.00103426 (0.131574)
0.4	0.00506541 (0.0595644)	-0.00164893 (0.103428)	0.00856499 (0.138227)	-0.00427827 (0.192773)
0.5	9.46511×10^{-6} (0.080832)	0.00143187 (0.140811)	0.00821915 (0.19256)	-0.00373388 (0.238901)
0.6	-0.00193361 (0.103372)	0.000510702 (0.170413)	0.00547577 (0.228127)	0.00388679 (0.289384)
0.7	-0.00413912 (0.119923)	-0.00132456 (0.202725)	0.00765032 (0.276109)	-0.00726503 (0.341083)
0.8	-0.00109334 (0.141536)	-0.00267304 (0.235878)	0.00131327 (0.315041)	-0.00319695 (0.396119)
0.9	-0.0056615 (0.169933)	-0.0036788 (0.267179)	0.000308504 (0.355221)	-0.00244952 (0.453053)

Table (2.11): The MSE of $\hat{\lambda}_{2\,SRS}(\hat{\lambda}^{*}_{2\,MERSS})$

ρ \ m	2	3	4	5
0.1	0.499551 (0.491253)	0.333345 (0.331635)	0.249649 (0.248509)	0.19864 (0.210132)
0.2	0.499826 (0.504642)	0.333675 (0.349085)	0.257373 (0.271994)	0.200124 (0.228403)
0.3	0.500996 (0.514995)	0.329593 (0.362998)	0.253507 (0.285276)	0.200313 (0.244522)
0.4	0.507055 (0.528374)	0.337384 (0.378078)	0.253354 (0.302438)	0.201426 (0.272699)
0.5	0.500087 (0.539565)	0.333134 (0.387088)	0.258028 (0.319192)	0.195592 (0.30074)
0.6	0.498256 (0.543571)	0.333864 (0.397733)	0.251646 (0.333728)	0.207673 (0.330537)
0.7	0.501187 (0.545423)	0.331669 (0.403483)	0.259182 (0.350917)	0.199612 (0.350742)
0.8	0.499357 (0.54799)	0.33138 (0.415925)	0.253013 (0.374888)	0.202879 (0.385361)
0.9	0.495889 (0.548266)	0.329498 (0.424497)	0.242145 (0.384798)	0.197268 (0.426565)

Table (2.12): The efficiency of $\hat{\lambda}^{*}_{2\,MERSS}$ w.r.t $\hat{\lambda}_{2\,SRS}$

ρ \ m	2	3	4	5
0.1	1.01689	1.00516	1.00458	0.945314
0.2	0.990457	0.955856	0.946246	0.876186
0.3	0.972817	0.907974	0.888635	0.8192
0.4	0.959652	0.892367	0.837706	0.738638
0.5	0.926834	0.860615	0.808381	0.650368
0.6	0.916634	0.839418	0.754045	0.628289
0.7	0.912654	0.822014	0.738587	0.569113
0.8	0.911251	0.79673	0.674904	0.526466
0.9	0.904468	0.776207	0.629278	0.462457

Based on Tables (2.10, 2.11 and 2.12) we conclude the following:

1- The absolute value of bias ($\hat{\lambda}_{2\,SRS}$) is increasing in the set size $m \geq 3$ for fixed ρ.

2- The value of MSE ($\hat{\lambda}_{2\,SRS}$) is decreasing in the set size m for fixed ρ.

3- There is no specific pattern for the absolute value of bias ($\hat{\lambda}^{*}_{2\,MERSS}$).

4- The value of MSE ($\hat{\lambda}^{*}_{2\,MERSS}$) is decreasing in the set size m for $\rho \geq 0.5$.

5- SRS more efficient than MERSS for $\rho \geq 0.2$.

8. Concluding Remarks

Based on the previous results obtained in this chapter, we can conclude that moving extreme ranked set sampling with concomitant variable gives more efficient estimators for the parameters of Downton's bivariate exponential distribution than the corresponding ones using simple random sample when other parameters are known.

CHAPTER THREE

Estimation of the Parameters of Downton's Bivariate Exponential Distribution Using the Method of Maximum Likelihood Estimation Based on MERSS

1. Introduction

In this chapter, we are interested in estimation of the parameters of DBED $(\lambda_1, \lambda_2, \rho)$ by using the method of maximum likelihood estimation (MLE) based on MERSS and compare these estimators to the corresponding ones using SRS. The likelihood function based on MERSS is given by following:

$$W(\gamma) = \prod_{k=1}^{m} f_{X_{(k:k)},Y_{[k:k]}}(x_{(k:k)}, y_{[k:k]}; \lambda_1, \lambda_2, \rho),$$

where $\gamma = (\lambda_1, \lambda_2, \rho)$ is a vector of unknown parameters.

The maximum likelihood method for estimating the parameters of DBED $(\lambda_1, \lambda_2, \rho)$ requires maximization of the likelihood $W(\gamma)$. Unfortunately; maximizing this function analytically is hard, because of the complexity of $f_{X_{(k:k)},Y_{[k:k]}}(x_{(k:k)}, y_{[k:k]}; \lambda_1, \lambda_2, \rho)$, therefore; in this situation we consider numerical methods. The asymptotic efficiency of the MLE based on MERSS with respect to MLE based on SRS, is obtained.

2. Fisher Information Matrix and Asymptotic Efficiency

The asymptotic efficiency of the MLE's of parameters, under some regularity conditions, is computed from the inverse of the Fisher information matrix. As n goes to infinity, the distribution of the MLE, $\hat{\theta}_{MLE}$, is asymptotically unbiased and also asymptotically normal with mean θ and var-cov matrix equal to the inverse of the Fisher information matrix, *i.e.*, $\hat{\theta}_{MLE} \xrightarrow{D} N(\theta, I^{-1}(\theta))$ (see **Myung and Navarro**

(2005)). So $Aeff\left(\hat{\theta}_{1MLE};\hat{\theta}_{2MLE}\right)=\frac{I_2^{-1}(i,i)}{I_1^{-1}(i,i)}$ where $I^{-1}(i,i)$ represents the i^{th} diagonal element in the inverse of the Fisher information matrix. In order, to obtain the Fisher information of the Downton's bivariate exponential distribution we start with $W(\gamma)$. In particular, it is easier to work with $\log(W(\gamma))$ than with $W(\gamma)$, and we know that both $W(\gamma)$ and $\log(W(\gamma))$ have the same maximizing value. Thus, in our case let W^* denote the log-likelihood function ...

$$W^* = \log f_{X_{(k:k)},Y_{[k:k]}}(x,y;\lambda_1,\lambda_2,\rho)$$

$$= \log\left\{ f_{X,Y}(x,y)k(1-\exp\left(-\frac{x}{\lambda_1}\right))^{k-1}\right\}$$

$$= \log\left\{\frac{1}{\lambda_1\lambda_2(1-\rho)}\exp\left[-\left(\frac{x}{\lambda_1(1-\rho)}+\frac{y}{\lambda_2(1-\rho)}\right)\right]\times I_0\left[\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right]\times k(1-\exp\left(-\frac{x}{\lambda_1}\right))^{k-1}\right.$$

$$= -\log(\lambda_1\lambda_2(1-\rho)) - \left(\frac{x}{\lambda_1(1-\rho)}+\frac{y}{\lambda_2(1-\rho)}\right) + \log I_0\left[\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right] + \log k + (k-1)\log\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right) \quad (1)$$

$$= L + N$$

where,

$$L = -\log(\lambda_1\lambda_2(1-\rho)) - \left(\frac{x}{\lambda_1(1-\rho)}+\frac{y}{\lambda_2(1-\rho)}\right) + \log I_0\left[\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right]$$

and

$$N = \log k + (k-1)\log\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right)$$

Also,

$I_1(z) = I_0'(z) = \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+1} \Big/ k!(k+1)!$, $z = \frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}$ (See Shi and Lai (1998)) (Note that what was written by Shi and Lai (1998) has a misprint in the summation that goes from 1 to ∞ where it must be from 0 to ∞)

$$I_2(z) = I_1'(z) = \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^{2k+2} \Big/ k!(k+2)!$$

In our case, based on Shi and Lai (1998), we get the following:

$$\frac{\partial W^*}{\partial \lambda_1} = -\frac{1}{\lambda_1} + \frac{x}{\lambda_1^2(1-\rho)} - \frac{(\rho xy)^{1/2}}{\lambda_1(\lambda_1\lambda_2)^{1/2}(1-\rho)} I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) I_0^{-1}\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) + (k-1)\frac{-\left(\frac{x}{\lambda_1^2}\right)\exp\left(-\frac{x}{\lambda_1}\right)}{1-\exp\left(-\frac{x}{\lambda_1}\right)}$$

$$\frac{\partial W^*}{\partial \lambda_2} = -\frac{1}{\lambda_2} + \frac{y}{\lambda_2^2(1-\rho)} - \frac{(\rho xy)^{1/2}}{\lambda_2(\lambda_1\lambda_2)^{1/2}(1-\rho)} I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) I_0^{-1}\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right).$$

$$\frac{\partial W^*}{\partial \rho} = \frac{1}{(1-\rho)} - \frac{x}{\lambda_1(1-\rho)^2} - \frac{y}{\lambda_2(1-\rho)^2} + \frac{1+\rho}{\rho(1-\rho)}\left(\frac{(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) I_0^{-1}\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho}\right.$$

Also,

$$\frac{\partial W^*}{\partial \lambda_1 \partial \lambda 2} = \frac{3(\rho xy)^{1/2} I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}{2\lambda_2^2(\lambda_1\lambda_2)^{1/2}(1-\rho) I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)} - \frac{(\rho xy) I_1^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}{\lambda_1\lambda_2^3(1-\rho)^2 I_0^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$+ \frac{(\rho xy) I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) + \rho xy) I_2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}{2\lambda_1\lambda_2^3(1-\rho)^2 I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$\frac{\partial W^*}{\partial \lambda_1 \partial \rho} = \frac{x}{\lambda_1^2(1-\rho)^2} + \frac{I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\left(-(xy)(1-\rho) - 2(\rho xy)^{1/2}\right)\left[I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) + I_2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\right.}{2\lambda_1(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2} I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$- \frac{\rho xy\left(\frac{xy}{(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}} + \frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)^2}\right)\left[I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right) + I_2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\right]}{2\lambda_1(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2} I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$+ \frac{(\rho xy)^{1/2}\left(\frac{xy}{(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}} + \frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)^2}\right) I_1^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}{\lambda_1(\lambda_1\lambda_2)^{1/2}(1-\rho) I_0^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$\frac{\partial W^{*}}{\partial\lambda_2\partial\rho}=\frac{y}{\lambda_2^2(1-\rho)^2}+\frac{I_1\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\left(-(xy)(1-\rho)-2(\rho xy)^{1/2}\right)\left[I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)+I_2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\right]}{2\lambda_2(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$-\frac{\rho xy\left(\frac{xy}{(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}}+\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)^2}\right)\left[I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)+I_2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)\right]}{2\lambda_2(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}I_0\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}$$

$$+\frac{(\rho xy)^{1/2}\left(\frac{xy}{(\lambda_1\lambda_2)^{1/2}(1-\rho)(\rho xy)^{1/2}}+\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)^2}\right)I_1^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}{\lambda_2(\lambda_1\lambda_2)^{1/2}(1-\rho)I_0^2\left(\frac{2(\rho xy)^{1/2}}{(\lambda_1\lambda_2)^{1/2}(1-\rho)}\right)}.$$

Now,

$$E\left(\frac{\partial W^{*}}{\partial\lambda_1}\right)^2=E\left(\frac{\partial L}{\partial\lambda_1}\right)^2+E\left(\frac{\partial N}{\partial\lambda_1}\right)^2.$$

But,

$$\frac{\partial N}{\partial\lambda_1}=(k-1)\frac{F'(x)}{F(x)},$$

Where

$$F(x)=1-\exp\left(-\frac{x}{\lambda_1}\right).$$

And

$$F'(x)=\frac{\partial F(x)}{\partial\lambda_1}=-\left(\frac{x}{\lambda_1^2}\right)\exp\left(-\frac{x}{\lambda_1}\right).$$

Thus,

$$E\left(\frac{\partial N}{\partial\lambda_1}\right)^2=E\left((k-1)\frac{F'(x)}{F(x)}\right)^2=(k-1)^2E\left(\frac{F'(x)}{F(x)}\right)^2.$$

It follows that

$$E\left(\frac{\partial N}{\partial\lambda_1}\right)^2=(k-1)^2\left[\int_0^\infty\left(\frac{F'(x)}{F(x)}\right)^2 f(X_{(k:k)})dx\right].$$

Substituting the values of F, F' and $f(X_{(k:k)})$ in the previous equation we get

$$B_k = E\left(\frac{\partial N}{\partial \lambda_1}\right)^2 = (k-1)^2\left[\int_0^\infty \left(\frac{-\left(\frac{x}{\lambda_1^2}\right)\exp\left(-\frac{x}{\lambda_1}\right)}{1-\exp\left(-\frac{x}{\lambda_1}\right)}\right)^2 \left(k\left(1-\exp\left(-\frac{x}{\lambda_1}\right)\right)^{k-1}\frac{1}{\lambda_1}\exp\left(-\frac{x}{\lambda_1}\right)\right)dx\right].$$

Thus, we can numerically find the values of B_k at different $k's$, the following table contains these values:

Table (3.1): Values of B_k for different $k's$

k	B_k
1	0
2	0.308228
3	0.888889
4	1.54167
5	2.20593

Therefore, it is easy to get the following table which contains the values of $\sum_{k=1}^{m} B_k$ for different set size m.

Table (3.2): Values of $\sum_{k=1}^{m} B_k$ for different m

m	B^*
2	0.308228
3	1.197117
4	2.738787
5	4.944717

Thus,

$$I_{11} = \sum_{k=1}^{m} E\left(\frac{\partial W^*}{\partial \lambda_1}\right)^2 = \sum_{k=1}^{m} E\left(\frac{\partial L}{\partial \lambda_1}\right)^2 + \sum_{k=1}^{m} B_k \,. \qquad (2)$$

But,

$$\sum_{k=1}^{m} E\left(\frac{\partial L}{\partial \lambda_1}\right)^2 = \sum_{k=1}^{m} \int_0^\infty \int_0^\infty \left(\frac{\partial L}{\partial \lambda_1}\right)^2 f_{X_{(k:k)},Y_{[k:k]}}(x,y)dxdy$$

$$= \int_0^\infty \int_0^\infty \left(\frac{\partial L}{\partial \lambda_1}\right)^2 f_{X,Y}(x,y) \sum_{k=1}^{m} k(F(x))^{k-1} dxdy \,.$$

Also,

$$\sum_{k=1}^{m} k(F(x))^{k-1} = \frac{\partial}{\partial F} \sum_{k=1}^{m} (F(x))^k = \frac{\partial}{\partial F}\left(\frac{(F(x))^{m+1} - F(x)}{F(x)-1}\right)$$

$$= \frac{(F(x)-1)((m+1)(F(x))^m - 1) - ((F(x))^{m+1} - F(x))}{(F(x)-1)^2}$$

$$= \frac{m(F(x))^{m+1} - (m+1)(F(x))^m + 1}{(F(x)-1)^2}, \qquad (3)$$

Using (2), (3) we get

$$I_{11} = \int_0^\infty \int_0^\infty \left(\frac{\partial L}{\partial \lambda_1}\right)^2 \frac{(F(x)-1)((m+1)(F(x))^m - 1) - ((F(x))^{m+1} - F(x))}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy + \sum_{k=1}^{m} B_k$$

Now to obtain I_{22},

$$E\left(\frac{\partial W^*}{\partial \lambda_2}\right)^2 = E\left(\frac{\partial L}{\partial \lambda_2}\right)^2 + E\left(\frac{\partial N}{\partial \lambda_2}\right)^2 = E\left(\frac{\partial L}{\partial \lambda_2}\right)^2 + Zero.$$

Thus,

$$I_{22} = \int_0^\infty \int_0^\infty \left(\frac{\partial L}{\partial \lambda_2}\right)^2 \frac{m(F(x))^{m+1} - (m+1)(F(x))^m + 1}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy \,. \qquad (4)$$

Also to obtain I_{12},

$$I_{12} = \sum_{k=1}^{m} E\left(\frac{\partial W^*}{\partial \lambda_1 \partial \lambda_2}\right)$$

So,

$$I_{12} = \int_0^\infty\int_0^\infty \left(\frac{\partial W^*}{\partial\lambda_1\partial\lambda_2}\right)\frac{m(F(x))^{m+1}-(m+1)(F(x))^m+1}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy \,. \tag{5}$$

Similarly to obtain I_{13},

$$I_{23} = \sum_{k-1}^{m} E\left(\frac{\partial W^*}{\partial\lambda_1\partial\rho}\right) = \int_0^\infty\int_0^\infty \left(\frac{\partial W^*}{\partial\lambda_1\partial\rho}\right)\frac{m(F(x))^{m+1}-(m+1)(F(x))^m+1}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy \,. \tag{6}$$

Also to obtain I_{23},

$$I_{23} = \sum_{k=1}^{m} E\left(\frac{\partial W^*}{\partial\lambda_2\partial\rho}\right) = \sum_{k=1}^{m} E\left(\frac{\partial L}{\partial\lambda_2\partial\rho}\right) + \sum_{k=1}^{m} E\left(\frac{\partial N}{\partial\lambda_2\partial\rho}\right)$$

$$= \int_0^\infty\int_0^\infty \left(\frac{\partial W^*}{\partial\lambda_1\partial\rho}\right)\frac{m(F(x))^{m+1}-(m+1)(F(x))^m+1}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy \,. \tag{7}$$

Finally, to obtain I_{33},

$$I_{33} = \sum_{k=1}^{m} E\left(\frac{\partial W^*}{\partial\rho}\right)^2 = \sum_{k=1}^{m} E\left(\frac{\partial L}{\partial\rho}\right)^2 + \sum_{k=1}^{m} E\left(\frac{\partial N}{\partial\rho}\right)^2$$

$$= \sum_{k=1}^{m} E\left(\frac{\partial L}{\partial\rho}\right)^2 + Zero$$

$$= \int_0^\infty\int_0^\infty \left(\frac{\partial L}{\partial\rho}\right)^2 \frac{m(F(x))^{m+1}-(m+1)(F(x))^m+1}{(F(x)-1)^2} f_{X,Y}(x,y)dxdy \,. \tag{8}$$

Thus, the information matrix and its inverse are

$$I_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \text{ and } I^{-1}_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} I^*_{11} & I^*_{12} & I^*_{13} \\ I^*_{21} & I^*_{22} & I^*_{23} \\ I^*_{31} & I^*_{32} & I^*_{33} \end{bmatrix}$$

where,

$I^*_{11} = (I_{22}I_{33} - I_{23}I_{32})/D;$ $\qquad I^*_{12} = -(I_{12}I_{23} - I_{13}I_{32})/D;$

$I^*_{13} = (I_{12}I_{23} - I_{13}I_{22})/D;$ $\qquad I^*_{22} = (I_{11}I_{33} - I_{13}I_{31})/D;$

$I^*_{23} = -(I_{11}I_{23} - I_{13}I_{21})/D;$ $\qquad I^*_{11} = (I_{11}I_{22} - I_{12}I_{21})/D;$

where,

$$D = I_{11}(I_{22}I_{33} - I_{23}I_{32}) - I_{12}(I_{21}I_{33} - I_{23}I_{31}) + I_{13}(I_{21}I_{32} - I_{22}I_{31}).$$

We want to find the asymptotic efficiency, to do so, we need to calculate $I^{-1}{}_{MERSS}(\lambda_1,\lambda_2,\rho)$ and $I^{-1}{}_{SRS}(\lambda_1,\lambda_2,\rho)$.

For example, if $\rho = 0.1$ and $m = 2$, then

$$I_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 2.85117 & 0.234834 & 0.216227 \\ 0.234834 & 2.04294 & -0.339328 \\ 0.216227 & -0.339328 & 2.02549 \end{bmatrix},$$

$$I^{-1}{}_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 0.358282 & -0.0488976 & -0.0464394 \\ -0.0488976 & 0.0510174 & 0.0906888 \\ -0.0464394 & 0.0906888 & 0.513858 \end{bmatrix}.$$

Also, By Shi and Lai (1998),

$$I_{SRS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 2.0318 & -0.1904 & -0.14188 \\ -1.1904 & 2.0318 & -0.14188 \\ 0.14188 & -0.14188 & 1.7342 \end{bmatrix},$$

$$I^{-1}{}_{SRS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 0.4999 & 0.04999 & -0.0464394 \\ 0.04999 & 0.5 & 0.0906888 \\ 0.04999 & 0.04999 & 0.548 \end{bmatrix}.$$

If $\rho = 0.9$ and $m = 5$, then

$$I_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 50.587 & 39.7937 & 29.9042 \\ 39.7937 & 43.5056 & -7.0958 \\ 29.9042 & -7.0958 & 261.303 \end{bmatrix},$$

$$I^{-1}{}_{MERSS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 0.109928 & -0.103057 & -0.0153789 \\ -0.103057 & 0.119703 & 0.0150447 \\ -0.0153789 & 0.0150447 & 0.00599552 \end{bmatrix}.$$

Also, By Shi and Lai (1998)

$$I_{SRS}(\lambda_1,\lambda_2,\rho) = \begin{bmatrix} 26.8482 & -23.1518 & -11.2408 \\ -23.1518 & 26.8482 & -11.2408 \\ -11.2408 & -11.2408 & 237.305 \end{bmatrix},$$

$$I^{-1}{}_{SRS}(\lambda_1, \lambda_2, \rho) = \begin{bmatrix} 0.2 & 0.18 & 0.018 \\ 0.18 & 0.2 & 0.018 \\ 0.018 & 0.018 & 0.00591925 \end{bmatrix}.$$

Table (3.3) contains the Asymptotic efficiency of the MLE using MERSS *w.r.t* the MLE using SRS.

Table (3.3) Asymptotic efficiency of the MLE of $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\rho}$ using MERSS *w.r.t* the MLE using SRS $Aeff(1)$, $Aeff(2)$ and $Aeff(3)$, respectively.

ρ	m	$Aeff(1)$	$Aeff(2)$	$Aeff(3)$
0.1	2	1.39555	0.98006	1.13653
	3	1.74488	0.90443	1.22220
	4	2.06092	0.82770	1.28289
	5	2.35068	0.76106	1.32896
0.2	2	1.38428	1.01723	1.11380
	3	1.70152	0.95447	1.17305
	4	1.97965	0.88414	1.21357
	5	2.22875	0.82137	1.24456
0.3	2	1.37993	1.06299	1.10701
	3	1.66157	1.01664	1.14524
	4	1.89947	0.95662	1.16983
	5	2.10757	0.90092	1.18890
0.4	2	1.38666	1.11896	1.11372
	3	1.63456	1.09152	1.13273
	4	1.83551	1.04416	1.14167
	5	2.00708	0.99715	1.14791
0.5	2	1.40688	1.18673	1.13302
	3	1.62480	1.18019	1.13273
	4	1.79286	1.14678	1.12430
	5	1.93162	1.10886	1.11501

0.6	2	1.44241	1.26806	1.16461
	3	1.63426	1.25022	1.14339
	4	1.77221	1.24462	1.11452
	5	1.87954	1.23494	1.08602
0.7	2	1.49477	1.46880	1.20833
	3	1.66344	1.43378	1.16287
	4	1.77190	1.39956	1.10926
	5	1.84638	1.36319	1.05727
0.8	2	1.56610	1.53558	1.26391
	3	1.71285	1.52146	1.18885
	4	1.79012	1.50174	1.10527
	5	1.82789	1.48033	1.02549
0.9	2	1.65876	1.62635	1.33022
	3	1.78229	1.60730	1.21748
	4	1.82415	1.57385	1.09827
	5	1.84563	1.55008	0.98728

Based on the previous table we conclude the following:

1. $Aeff(\hat{\lambda}_{1\,MERSS,MLE}, \hat{\lambda}_{1\,SRS,MLE})$ is decreasing in $\rho \le 0.3$ for fixed set size and increasing in $\rho \ge 0.4$ for fixed set size m.

2. $Aeff(\hat{\lambda}_{1\,MERSS,MLE}, \hat{\lambda}_{1\,SRS,MLE})$ is larger than 1 and increasing in the set size m for fixed ρ.

3. $Aeff(\hat{\lambda}_{2\,MERSS,MLE}, \hat{\lambda}_{2\,SRS,MLE})$ is larger than 1 for $\rho \ge 0.5$ and increasing in ρ for fixed set size m.

4. $Aeff(\hat{\lambda}_{2\,MERSS,MLE}, \hat{\lambda}_{2\,SRS,MLE})$ is decreasing in the set size m for fixed ρ.

5. $Aeff(\hat{\rho}_{MERSS,MLE}, \hat{\rho}_{SRS,MLE})$ is increasing in $\rho \ge 0.4$ for fixed set size m.

6. $Aeff(\hat{\rho}_{MERSS,MLE}, \hat{\rho}_{SRS,MLE})$ is increasing in the set size m for $\rho \leq 0.4$ and decreasing in the set size m for $\rho \geq 0.5$.

3. Concluding Remarks

Based on the previous results obtained in this chapter, we can conclude that some of the estimators obtained using maximum likelihood estimator based on moving extreme ranked set sampling with concomitant variable gives asymptotically more efficient estimators for the parameters of Downton's bivariate exponential distribution than the corresponding ones using simple random sample.

CHAPTER FOUR

CONCLUTIONS AND SOME SUGGESTED FURTHER WORK

This chapter concludes the thesis. A summary of the conclusions is presented in Section 1, while Section 2 contains some suggestions for future work.

1. Conclusions

Moving extreme ranked set sampling is a useful variation of ranked set sampling and more applicable since it allows for an increase of set size without introducing extra ranking errors. In this procedure, only two extreme values (maximum or minimum) of sets of varied size was identified (by judgment) for quantification. In this thesis, the main goal was to estimate the parameters of Downton's bivariate exponential distribution using MERSS with concomitant variable. It was assumed that X can be ranked visually while Y was highly correlated with X. It was shown that the use of MERSS with concomitant variable gives more efficient estimators for the parameters of Downton's bivariate exponential distribution than the corresponding ones using Simple Random Sample. In addition, we derive the best linear unbiased estimators of λ_1 and λ_2. It was shown that these estimators were very close to the corresponding naive estimators. Moreover, Fisher information matrix of Downton's bivariate exponential distribution was derived and used to find the asymptotic efficiency of the maximum likelihood estimator of each of the parameters using MERSS with respect to those based on SRS. It was shown that some of the estimators obtained using MLE based on MERSS were asymptotically more efficient than the corresponding ones based on SRS.

2. Some Suggested Further Works

There are other situations where MERSS can be used. The following are some suggested further work on this topic.

- Estimation of the parameters of other bivariate distributions such as Morgenstern type bivariate exponential distribution using MERSS.
- Estimation of the parameters of Downton's bivariate exponential distribution using double or multistage MERSS.
- Estimation of the parameters of Downton's bivariate exponential distribution using bivariate MERSS.
- Testing hypothesis about the parameters of Downton's bivariate exponential distribution using MERSS.
- Bayesian estimation of the parameters of Downton's bivariate exponential distribution using MERSS.

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